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A Qualitative-Structural Approach to the Modeling of Knowledge

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Summary

During the last decade, Doignon & Falmagne have developed a qualitative structural approach to the modeling of knowledge. This *knowledge structures theory* involves several noteworthy advantages – the most important is the economical and efficient assessment of individual knowledge states within a specified field of knowledge. However, the knowledge structures theory suffers from a purely descriptive approach to the representation of knowledge that is without any cognitive interpretation. Recent efforts to base knowledge structures on underlying skill sets appear not to be appropriate to enrich the behavioral approach cognitively because they treat skills as pure epiphenomena of empirically constituted knowledge structures.

The paper presented here proposes a way in which the behavioral knowledge structure approach could be reconciled with traditional and cognitive explanatory features of knowledge assessment. It develops a formal framework in terms of *competence* and *performance* where *performance* is conceived as the observable solution behavior of a person on a set of domain-specific problems, and *competence* is understood as a theoretical construct for explaining and predicting performance. A basic presupposition is that existing domain-specific theories can be utilized for a genuine modeling of theoretically meaningful competence (skills) structures, whereas performance structures are goal-directedly constructed, structurally adequate empirical representations of competence modelings.

The central concept of our *competence-performance approach* to knowledge modeling is a mathematical structure termed a *diagnostic* that creates a relationship between a family of *competence states* and a family of *performance states*. In an *order-stable diagnostic*, the family of competence states as well as the family of performance states are conceptualized as partially ordered sets and there exists an order-preserving (montone) function that maps the family of competence states *onto* the family of performance states. In a *union-stable diagnostic*, both the family of competence states and the family of performance states are union-stable and there exists a union-preserving function from the family of competence states *onto* the family of performance states.

In this paper we will derive conditions and several properties of order-stable and union-stable diagnostics. The main focus hereby is on the concept of a *union-stable diagnostic* because this concept includes the extension and reinterpretation of the central concept of a "knowledge space" in Doignon & Falmagne's theory. We will then report on an empirical investigation which illustrates some advantages of union-stable diagnostics in practical application. The paper concludes with a summary and general discussion on the proposed competence-performance approach to the knowledge structures theory.

Key words: competence–performance, knowledge structures, knowledge spaces, skills

Zusammenfassung

In den vergangenen zehn Jahren haben Doignon & Falmagne mit der *Wissensstruktur-Theorie* einen qualitativ-strukturellen Ansatz zur Wissensmodellierung entwickelt. Diese Theorie zeichnet sich durch einige bemerkenswerte Vorzüge aus; vor allem ermöglicht sie die ökonomische und effiziente Diagnose individuellen Wissens innerhalb eines spezifischen Wissensbereichs. Allerdings ist unter kognitiven Gesichtspunkten der rein deskriptive Ansatz der Wissensstruktur-Theorie als gravierendes Defizit anzusehen. Jüngste Bemühungen, Wissensstrukturen auf zugrundeliegenden Mengen von "skills" zu basieren, erscheinen zur kognitiven Anreicherung des behavioralen Ansatzes ungeeignet, weil die skills dabei lediglich als Epiphänomene der empirisch konstituierten Wissensstrukturen fungieren.

Der vorliegende Beitrag beschreibt in Begriffen von *Kompetenz* und *Performanz* einen erweiterten formalen Rahmen für die Wissensstruktur-Theorie, um diese an traditionelle und an kognitive Erklärungsansätze der Wissensdiagnostik anzubinden. *Performanz* wird hier aufgefaßt wird als beobachtbares Verhalten einer Person beim Lösen spezifischer Aufgaben aus einer Wissensdomäne, während *Kompetenz* als ein theoretisches Konstrukt zur Erklärung und Prognose von Performanz verstanden wird. Dabei beruht der Ansatz auf der grundlegenden Annahme, daß verfügbare domänspezifische Theorien für eine genuine Modellierung theoretisch bedeutungshaltiger Kompetenzstrukturen genutzt werden können, während Performanzstrukturen zielorientiert als strukturadäquate empirische Repräsentationen von Kompetenzmodellierungen konstruiert werden.

Das zentrale Konzept des *Kompetenz-Performanz-Ansatzes* für die Wissensmodellierung ist eine mathematische Struktur, bezeichnet als *Diagnostik*, welche eine Beziehung zwischen einer Familie von *Kompetenzzuständen* und einer Familie von *Performanzzuständen* herstellt. In einer *ordnungstreuen Diagnostik* sind die Familien der Kompetenzzustände und der Performanzzustände als partielle Ordnungen konzeptualisiert, und die Familie der Kompetenzzustände wird durch eine ordnungserhaltende (monotone) Funktion *auf* die Familie der Performanzzustände abgebildet. In einer *vereinigungstreuen Diagnostik* sind die Familien der Kompetenzzustände und der Performanzzustände jeweils vereinigungsstabil, und es existiert eine vereinigungstreue Abbildung der Familie der Kompetenzzustände *auf* die der Performanzzustände.

Im Laufe des vorliegenden Beitrages werden Bedingungen und einige Eigenschaften ordnungstreuer und vereinigungstreuer Diagnostiken hergeleitet. Der Fokus liegt dabei auf dem Konzept der *vereinigungstreuen Diagnostik*, da dieses Konzept die Erweiterung und Reinterpretation des in der Theorie von Doignon & Falmagne zentralen Konzepts des "Wissensraumes" darstellt. Berichtet wird weiterhin über eine empirische Untersuchung, welche einige Vorzüge vereinigungstreuer Diagnostiken in der praktischen Anwendung illustriert. Der Beitrag schließt mit einer Zusammenfassung und allgemeinen Diskussion über den vorgeschlagenen Kompetenz-Performanz-Ansatz für die Wissensstruktur-Theorie.

Schlüsselwörter: Kompetenz–Performanz, Wissensstrukturen, Wissensräume, skills

1 Introduction

Approaches to knowledge representation

During the past three decades, cognitive psychologists have developed several powerful systems for the representation of qualitative-structural aspects of knowledge in human memory, for example propositional and schema-based representation systems, mental models, neuronal networks etc. (for an overview see e.g. Mandl & Spada, 1988). In contrast to the research on Artificial Intelligence, the central criterion for the usefulness of a particular psychological knowledge representation model is the empirical (psychological) validity of that model (Spada & Mandl, 1988). Validity becomes crucial, for instance, when the model is to be used as a frame of reference for diagnosing individual knowledge and/or goal-directed adaptive instruction.

Unfortunately, many of the currently existing cognitive theories and knowledge representation modelings are related too poorly and indirectly to behavioral data because they are formulated to a great extent as computerized systems on a level that Anderson (1990) calls the *implementation level*. In order to bridge the gap between the highly elaborated cognitive modelings and the level of observation, many cognitive psychologists propagate successful computer simulation of observable behavior as an indication for the psychological meaningfulness of the underlying knowledge representation model (e.g. Spada & Reimann, 1988). But there is at least one specific methodological problem with simulations, the *irrelevant-specification problem* (Newell, 1990), concerning the suspicion that computer simulations stipulate a large number of assumptions, "just to make the simulation run" (Reitman, 1985, cit. Newell, 1990, p.23). The difficulty then is to separate the psychologically relevant claims from the adventitious aspects of the program[1]. Therefore, computer simulation as a method of validating a cognitive theory should be judged sceptically until at least the basic assumptions for the implemented cognitive model have been empirically proved (see e.g. Mandl, Friedrich & Hron, 1988).

The deficiency of diagnostical instruments for relating cognitive theories to empirical data (Tergan, 1986), or, stated another way, the discrepancy between the richness of cognitive modeling approaches and the relative paucity of observable behavior data continues to generate a serious handicap for a sufficient psychological validation of the theoretical models as well as for practical applications of these models in the context of a theory-oriented qualitative knowledge diagnosing.

[1] Newell (1990) himself argues that a "unified theory of cognition" would contribute to solving this problem.

An approach to knowledge representation and diagnosis of a completely different type is Doignon & Falmagne's *knowledge structures theory* developed during the last decade. The major advantage of Doignon & Falmagne's theory is that knowledge representation and knowledge diagnosis are closely related on the level of behavioral data. However, in marked contrast to the problems of many existing cognitive theories and systems for knowledge representation, a serious deficit of the knowledge structures conception is the lack of theoretical foundation for the operationally constituted knowledge structures. Nevertheless, despite its disadvantages, the knowledge structures theory seems able to contribute some ideas to the issues of validating models of knowledge representation and of diagnosing individual knowledge states. For this, however, Doignon & Falmagne's original approach has to be extended in several ways.

This paper attempts to take steps toward linking the knowledge structures theory to the psychometrical tradition and to contemporary cognitive psychology. In the next two subsections, Doignon & Falmagne's *theory of knowledge spaces* will be presented and briefly discussed, and the central ideas of a *competence-performance conception* for the knowledge structures theory will be outlined. In Section 2 the basics of the competence-performance conception will be introduced. The formal efforts of this paper will concentrate, in Sections 3 and 4, on an extension of the theory of knowledge spaces. Section 5 reports on a study concerning the newly introduced modeling approach for the theory of knowledge spaces. Some aspects of the competence-performance conception will be discussed in Section 6.

The knowledge structures theory

The *knowledge structures theory* (Doignon & Falmagne, 1985; see also Falmagne, Koppen, Villano, Johannesen & Doignon, 1990) presupposes that the "knowledge" of an individual in a particular domain of knowledge can be operationalized as the solving behavior of that individual on a domain-specific set X of *problems*. If the solution result is binarily coded by correct/incorrect, then the *knowledge state* of an individual in the given field of knowledge can be formally described as the subset of problems from X he/she is capable of solving. Now, it is a well-known fact that there often exist solution dependencies between problems within a certain field of knowledge. Within the knowledge structures theory, that fact is taken into account by two distinct conceptualizations.

The first approach for modeling the solution dependencies of a domain of knowledge is to assign to each problem $x \in X$ a family of subsets of X called *clauses*, with the interpretation that, if a person is capable of solving x then he/she is capable of solving all problems in at least one of these clauses. This idea is involved in the following notion of a *surmise system*. A *surmise system* is defined as a pair (X, σ) with a set X (of problems) and a mapping σ, called *surmise function on* X, that associates to each $x \in X$ a family $\sigma(x)$ of subsets of X called the *clauses* for x so that three conditions hold:

(1) Every clause for x contains x;

(2) If some clause C for x contains some problem $y \in X$, then there exists some clause D for y with $D \subseteq C$;

(3) Any clause C for x is minimal with respect to $\subseteq$.

The second approach is obvious. Because of the solution dependencies on the set X of problems, not each subset of X should indicate an observable knowledge state. Therefore, the set of all "empirically possible" knowledge states forms a particular family K of subsets of X which is, in general, not identical with the power set of X. The pair (X, K) is said to be a *knowledge structure.* Special interest is directed to *knowledge spaces.* A knowledge structure (X, K) is called a *knowledge space* when the set X and the empty set $\emptyset$ belong to K and K is stable under union. Motivation for this conceptualization can be found in Doignon & Falmagne (1985) or in Falmagne et al. (1990). One reason for the importance of knowledge spaces is that they can be efficiently stored in the form of a *basis.* The *basis* of a knowledge space (X, K) is the minimal subfamily $\mathcal{B}(K)$ of K so that each knowledge state in K can be written as a union of elements of $\mathcal{B}(K)$.

The two concepts of a knowledge space and of a surmise system play an essential role in Doignon & Falmagne's theory. The reason is that Doignon & Falmagne (1985) succeeded in constructing a one-to-one correspondence between surmise systems and knowledge spaces. For a given knowledge space (X, K), let K_x denote the family of states containing some problem $x \in X$, and let $\hat{K}_x :=$ Min K_x represent the set of all minimal states in K_x (with respect to the subset relation $\subseteq$). Taking into account that the families $\hat{K}_x$ $(x \in X)$ are closely related to the basis $\mathcal{B}(K)$ of the knowledge space (X, K) through $\mathcal{B}(K) = \bigcup\{\hat{K}_x \mid x \in X\}$, the following theorem is immediately comprehensible.

> **Theorem** (Doignon & Falmagne, 1985): Suppose that X is a nonempty, finite set of problems. Then there is a one-to-one correspondence between the set of all surmise systems (X, σ) on X and the set of all knowledge spaces (X, K) on X. This correspondence is specified by the equation $\sigma(x) = \hat{K}_x$.

With these formally equivalent concepts of a *knowledge space* and a *surmise system* Doignon & Falmagne capture those structures inherent in knowledge that can be operationalized as solvability relations between problems of a specified field of knowledge. The decisive advantage of this set-theoretical approach to knowledge representation is that domain-specific knowledge is modeled as a family of empirically expected states that can, in principle, be observed as solution patterns on problems. In this way, a knowledge representation model immediately provides a frame for individual knowledge diagnosing. Based on knowledge structures (once they have been empirically validated) automatic procedures for an efficient assessment of knowledge can be designed, and this is the essential aim of Doignon & Falmagne's knowledge structures theory (see Doignon & Falmagne, 1985; Falmagne & Doignon, 1988a, 1988b).

A major problem of the knowledge structures approach, however, is the fact that it yields purely behavioral and descriptive models of knowledge domains without any theoretical explanation for the observed knowledge states. Suppose, for example, a certain knowledge structure has been accepted as a valid description of empirically observable knowledge states – what about knowledge states based on another set of problems ? Without some theory on the underlying *skills* or *competencies* it is not possible to predict the problem solving behavior for a new problem; moreover, in connection with learning and instruction within a certain field of knowledge, the knowledge model does not provide any advice to help identify which kind of information a person should be taught in order to enable him/her to master a problem previously not solved.

Also in Doignon & Falmagne's research group the problems of a merely descriptive knowledge modeling seem to be recognized. Referring to a long-standing psychometric tradition, an approach for basing a knowledge structure on a family of "skills" is outlined in Falmagne et al. (1990). The central idea is that a person should master a given problem if he/she possesses the necessary skills (see also Marshall, 1981). This approach is developed further in Doignon (1994) and recently in Düntsch & Gediga (1995). In this line of research, however, skills are treated as pure epiphenomena of empirically constituted structures, as abstract entities, that only have to be properly assigned to a specified set of problems in order to formally generate the previously established knowledge structure or knowledge space (a typical question is whether any given knowledge structure or knowledge space can be generated from a minimal and unique skill assignment – see e.g. Doignon, 1994; Düntsch & Gediga, 1995). In this way, one merely obtains a set of uninterpreted "hidden factors" (Doignon, 1994) generating the particular empirical knowledge structure but no meaningful theory on the domain-specific skills underlying the solving-behavior on various samples of problem sets of that knowledge domain. This procedure may be indicated in preliminary studies of specific knowledge domains where no theoretical concepts of the domain-specific knowledge are available. In a growing number of fields, however, detailed theoretical models for the observable behavior in answering knowledge questions or solving certain types of problems have been developed.

A competence-performance approach

The purpose of this paper is to present an approach that conceptualizes a domain of knowledge in terms of *competence* and *performance. Performance* is conceived as the empirically observable solution behavior on a particular set of problems, *competence (skills, ability, knowledge)* as theoretical entities for the explanation and prediction of performance. Competence is assumed to be modeled as a genuine meaningful *competence structure* (resp. *competence ordering, competence space*), based in general, on empirical and theoretical results from scientific research in the field of interest. The following examples may concretize how domain-specific theories can be utilized for competence modeling.

EXAMPLE 1: Spada & Kluwe (1981) relate their study on balance beams problems to Piaget & Inhelder's research on the genesis of the concept of "proportion". In this case, a *developmental theory for the genesis of certain skills* (or "competencies") is utilized for the prediction of theoretically expectable performance states.

EXAMPLE 2: For the purpose of knowledge modeling in elementary physics, Opwis, Spada, Bellert & Schweizer (1994) use a *theory on multiple mental representation of physical knowledge.* Knowledge elements (formulated as production rules) on a qualitative-relational, a quantitative-relational, and a quantitative level are explicated. Learning is conceptualized as knowledge acquisition on one representational level or as a transition from one level to another. Then, the diagnosed individual knowledge can be conceived as a state (we would say a "competence state") consisting of a particular subset of knowledge elements. Note that, according to the theory, only certain subsets of knowledge elements are accepted as states.

EXAMPLE 3: Schrepp (1995) reports on an investigation where a *process model for interindividual differences in solving letter series completion problems* was to be tested. The model is an extension of a model described in Simon & Kotovsky (1963). It is formulated as an algorithm parametrized by three parameters that account for the individual capability of solving certain problems. By the model, the family of possible parameter combinations and, therefore, the set of empirically expectable solution patterns on the problems, are strongly restricted. That provides a strong empirical test for the underlying model. In our context, the family of theoretically admitted parameter combinations may be considered a family of theory-based "competence states".

EXAMPLE 4: *Curricular networks of learning/teaching goals* in a particular knowledge domain are used in Korossy (1993) for the establishment of "competence structures" and corresponding "performance structures". In Section 5 of this paper, we will present a reanalysis of a study reported in Korossy (1996). In this study, the competence modeling is essentially directed by curricular considerations on learning/teaching goals in a specified area of elementary geometry, and performance is established as a theory-based empirical representation of the competence model.

The essential distinction of the competence-performance approach to the "skill functions approach" is that competencies/skills are no longer viewed as epiphenomena of the observable problem solving behavior; instead, competence is given a priority role in knowledge modeling. Our concept of knowledge modeling is first to build a competence structure on the basis of a domain-specific theory, and only after that to construct appropriate structure-preserving performance representations of the given competence structure. As structures for modeling competence we concentrate here on those types of structures introduced in Doignon & Falmagne's knowledge structures theory.

The competence-performance approach to the knowledge structures theory is confronted with three main tasks:

(1) to define appropriate representation concepts for the mapping of the various types of competence structures to the level of performance;

(2) to explicate conditions for problem sets which guarantee structurally adequate empirical representations of the competence structures;

(3) to analyze properties and some critical situations emerging as a consequence of the defined representation concepts.

This article is concerned with tasks (1) and (2); task (3) is the subject of an another paper.

The next three sections introduce selected parts of a theoretical framework for competence-performance structures. In Section 5, an empirical application of the theory within a domain of elementary geometry is reported. Altogether, this contribution shall outline one possible approach in which Doignon & Falmagne's knowledge structures theory could be reconciled with the psychometric tradition.

2 The competence-performance approach

Basic assumptions

The basic premise is to make a clear distinction between *competence (skills, ability, knowledge)* as theoretical entities and *performance* as the empirically observable solution behavior on certain given problems. The theoretical approach to be outlined in this paper is based on the following assumptions.

2.1 **Assumptions:**

(1) A particular knowledge domain $\mathcal{W}$ can be modeled through a finite, non-empty family $\mathcal{K}$ of *"competence states"*. The *knowledge, capability* or the *set of skills* of a person with respect to the domain (at a given time) is a certain element of $\mathcal{K}$. The family $\mathcal{K}$ is conceived as a set of not directly observable theoretical constructs (ideally) constituted on some domain specific (psychological) theory.

(2) The family $\mathcal{K}$ of competence states may be structured through order relations or operations of supremum or infimum. Taken into special consideration is the case that the competence states themselves are specified as particular subsets of a domain-specific set $\mathcal{E}$ of so-called *"elementary competencies"*. The set $\mathcal{E}$ may be structured for its part (e.g. through surmise structures); these structures would then limit the family $\mathcal{K}$ of competence states.

(3) The knowledge domain $\mathcal{W}$ can also be modeled by a set $\mathcal{A}$ of *"problems"*. It is assumed that

   (a) each problem $x \in \mathcal{A}$ is solvable exclusively with the knowledge modeled by $\mathcal{K}$;

   (b) for each problem x and each competence state $\kappa \in \mathcal{K}$ it can uniquely be decided whether or not x can be solved in κ.

   It is explicitly taken into account that a problem may be solvable in various ways (in different competence states respectively).

(4) Every person is – according to his/her momentary competence state – capable of solving certain problems and only those problems. The result of this solving behavior is observable as a *"correct solution"* or an *"incorrect solution"* for each applied problem. The subset of correctly solved problems is called the *"solution pattern"* for that person.

(5) The solvability conditions of the problems in $\mathcal{K}$ provide theoretically expected solution patterns. A theoretically expected solution pattern is called a *"performance state"*. An empirical validation of the knowledge modeling means that the performance states coincide with the observed solution patterns. If the modeling is validated, the performance state of a person suggests conclusions on the possible competence state of that person.

With these assumptions in mind we reinterpret the basic concept of a knowledge structure in the following manner:

- A pair $(\mathcal{E}, \mathcal{K})$ consisting of a finite, non-empty set $\mathcal{E}$ of *elementary competencies* and a non-empty family $\mathcal{K}$ of subsets of $\mathcal{E}$ called *competence states* is said to be a *competence structure.* It is assumed that for each $\varepsilon \in \mathcal{E}$ there exists a competence state $\kappa \in \mathcal{K}$ such that $\varepsilon \in \kappa$.[2]

- A pair $(\mathcal{A}, \mathcal{P})$ consisting of a finite, non-empty set $\mathcal{A}$ of *problems* and a non-empty family $\mathcal{P}$ of subsets of $\mathcal{A}$ called *performance states* is said to be a *performance structure.* Again, it is assumed that for each $x \in \mathcal{A}$ there exists a performance state $Z \in \mathcal{P}$ such that $x \in Z$.

At this point, the concepts *competence structure* and *performance structure* are formally equivalent and only differ with respect to their psychological interpretation. However, with the definition of the concept of a "diagnostic" in the next subsection, these structures will also be given formally different positions.

The concept of a diagnostic

The basic concept of a *diagnostic* is developed in two steps. The first step introduces the concept of a "competence-based problem set"; the second step leads directly to the concept of a diagnostic.

The idea for the concept of a *competence-based problem set* is the following: A problem $x \in \mathcal{A}$ can be related to the set $\mathcal{K}$ of competence states or *"interpreted"* in $\mathcal{K}$, if $\mathcal{K}$ includes all competence states of the domain which are alternatively sufficient for solving that problem. Then, the *interpretation* of x in $\mathcal{K}$ can be realized through mapping x to a set

$$k(x) := \{\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_r\} \quad (r \in \mathbb{N}),$$

specific for x, which contains exactly those competence states of $\mathcal{K}$, in which x can be solved. This idea leads to the following definition.

[2]This postulate is equivalent to $\cup\mathcal{K} = \mathcal{E}$, so that $\mathcal{K}$ itself contains all information on $\mathcal{E}$.

2.2 **Definition**: Let $\mathcal{K}$ be a set of competence states. Further, let $\mathcal{A}$ be a set of problems and $k : \mathcal{A} \longrightarrow \wp(\mathcal{K})$ a mapping which assigns to each problem $x \in \mathcal{A}$ a subset $k_x := k(x) \subseteq \mathcal{K}$ of competence states (the set of competence states in each of which x is solvable) so that

(k1) $k_x \neq \emptyset$;

(k2) $k_x \neq \mathcal{K}$.

Then the problem set $\mathcal{A}$ is called a problem set *based on* $\mathcal{K}$ or *interpreted in* $\mathcal{K}$; k is called the (corresponding) *interpretation function*, and for each problem $x \in \mathcal{A}$ the set k_x is called the *interpretation of* x or the *problem concept of* x *in* $\mathcal{K}$.

Obviously, the above definition is in concordance with the idea of interpreting problems in a set of competence states. The existence (well-definedness) of the interpretation function assures that to each problem a unique interpretation *within* the considered set of competence states is assigned. The conditions $k(\mathcal{A}) \subseteq \wp(\mathcal{K})$ and (k1) mean that no *"practically relevant"* solving competencies should lie *outside* $\mathcal{K}$. The postulate $k_x \neq \mathcal{K}$ for each problem $x \in \mathcal{A}$ accounts for the idea that solving a problem seen as representative for the domain should imply some information on the given set of competence states[3].

Remark: Given a set $\mathcal{A}$ of problems interpreted in $\mathcal{K}$ through $k : \mathcal{A} \longrightarrow \wp(\mathcal{K})$, there is a uniquely determined *complementary function*

$$(2.3) \qquad k^- : \begin{cases} \mathcal{A} \longrightarrow \wp(\mathcal{K}) \\ x \longmapsto k_x^- := \mathcal{K} \setminus k_x = \{\mu \in \mathcal{K} \mid \mu \notin k_x\} \end{cases},$$

that assigns to each problem $x \in \mathcal{A}$ the *complementary concept* k_x^- including all those competence states in which x is *not* solvable. The function k^- will be used later.

Now, let $\mathcal{A}$ be a set of problems interpreted in a set of competence states $\mathcal{K}$ with an interpretation function k. A reasonable concept of a competence-based *performance state* should involve the following idea: A subset $Z \subseteq \mathcal{A}$ of problems is considered a *performance state*, when there exists a competence state $\kappa \in \mathcal{K}$ so that Z includes

[3]The definition in 2.2 corresponds to the concept of a "skill assignment" in Doignon (1994). If the competence states are specified as subsets of a set $\mathcal{E}$ of elementary competencies, then the interpretation function coincides with the notion of a "skill multiassignment" in Doignon (1994), but not exactly with the concept of a "skill function" in Düntsch & Gediga (1995) that is used to specify those sets of skills which are *minimally sufficient* to solve a problem x.

exactly those problems which are solvable in κ. This idea can be formalized through a mapping

$$(2.4) \qquad p : \begin{cases} \mathcal{K} \longrightarrow \wp(\mathcal{A}) \\ \kappa \longmapsto p(\kappa) := \{x \in \mathcal{A} \mid \kappa \in k_x\} \end{cases}$$

which assigns to each competence state $\kappa \in \mathcal{K}$ the unique (possibly empty) *set of all problems solvable in* κ; then each element in the image of $\mathcal{K}$ under p is a (*competence based*) *performance state* in the above sense, and the total set of all those performance states *"represents"* the given set of competence states $\mathcal{K}$, possibly as a "diminished" image of $\mathcal{K}$.

The mappings k and p are the tools for linking a performance structure $(\mathcal{A}, \mathcal{P})$ to a set of competence states.

2.5 **Definition:** Let $\mathcal{K}$ be a set of competence states and $(\mathcal{A}, \mathcal{P})$ a performance structure. Assume the following conditions hold:

(1) $\mathcal{A}$ is a problem set interpreted in $\mathcal{K}$ by a function $k : \mathcal{A} \longrightarrow \wp(\mathcal{K})$;

(2) the image of the mapping $p : \mathcal{K} \longrightarrow \wp(\mathcal{A})$ induced by k according to (2.4) is $p(\mathcal{K}) = \mathcal{P}$.

Then $(\mathcal{A}, \mathcal{P})$ is called a *representation of* $\mathcal{K}$ (under p). The function k is called the *interpretation function for* $(\mathcal{A}, \mathcal{P})$ (relative to $\mathcal{K}$), the mapping p is called the *representation function of* $\mathcal{K}$ (*relative to* $(\mathcal{A}, \mathcal{P})$). The 5–tupel $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is said to be a *diagnostic*. If the competence states of $\mathcal{K}$ are subsets of a set $\mathcal{E}$ of elementary competencies (that is, if $(\mathcal{E}, \mathcal{K})$ is a competence structure) then we usually denote the diagnostic as a 6–tupel $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$.

This definition of a diagnostic resp. a performance structure $(\mathcal{A}, \mathcal{P})$ as a representation of a set $\mathcal{K}$ of competence states under a representation function p reflects the previous characterization of (*competence based*) *performance states* in $\mathcal{P}$. Condition (2), namely that the image of $\mathcal{K}$ under the representation function p be *equal* to $\mathcal{P}$, formally assures that on the one hand all subsets of problems generated by p are members of the performance structure $(\mathcal{A}, \mathcal{P})$ (because $p(\mathcal{K}) \subseteq \mathcal{P}$ is postulated), and that, on the other hand, all performance states occurring in $\mathcal{P}$ are images of competence states in $\mathcal{K}$ and in that sense theoretically interpretable (because $p(\mathcal{K}) \supseteq \mathcal{P}$ is required).

Let us point here to a property inherent in the concept of a diagnostic that is advantageous for the

construction of a diagnostic:

For a given set $\mathcal{K}$ of competence states the construction of a representation can proceed *step by step.* If in a diagnostic $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ the set $\mathcal{A}$ of problems is extended to the set $\mathcal{A}' := \mathcal{A} \cup \{x\}$ by a problem x with interpretation in $\mathcal{K}$, then the resulting structure is again a diagnostic.

That is, depending on practical requirement, a representation of a given set of competence states can be established step by step becoming an increasingly more accurate image of the set of competence states.

For later use, two easily verified lemmata for diagnostics are mentioned here. The first lemma formulates a necessary and sufficient condition for the existence of an empty state in a representation of a competence structure.

2.6 **Lemma**: *For a diagnostic* $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *holds:* $\bigcup k(\mathcal{A}) = \mathcal{K} \iff \emptyset \notin \mathcal{P}$.

The following lemma states, that not only the interpretation function k determines uniquely the representation function but, vice versa, also the interpretation function is uniquely determined by the representation function p. This is in concordance with Düntsch & Gediga (1995, Proposition 2.3.). We will need this lemma repeatedly. The proof is obtained immediately by referring to the definition of p in (2.4).

2.7 **Lemma**: *Let* $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *be a diagnostic.*

(1) $x \in p(\kappa) \iff \kappa \in k_x$, *for all* $x \in \mathcal{A}$, $\kappa \in \mathcal{K}$.

(2) $k_x = \{\lambda \in \mathcal{K} \mid x \in p(\lambda)\}$, *for all* $x \in \mathcal{A}$.

In general, the concept of a diagnostic seems to fit the assumptions in 2.1 fairly well. The advantages of the proposed conceptualization are obvious: When a performance structure has been constructed as a representation of a well-founded domain-specific competence structure, then empirically observable solution patterns resp. solvability relations between problems can be theoretically explained and as empirical hypotheses derived and tested. More fundamentally, within this approach to knowledge conceptualization, a domain-specific theory has a chance for validation by way of modeling it as a competence model and representing it as a performance structure that can be tested directly. Whenever such a system of a competence and a performance structure (that is, a diagnostic) is accepted as psychologically valid, then an observed performance state of a person can be related back to the possibly underlying competence state of that person, and this competence diagnosis will be the more accurate the more precisely the performance structure represents the competence structure[4].

From a mathematical viewpoint the concept of a diagnostic is a rather weak concept. Nevertheless, it should be emphasized that for several modeling tasks the weak concept of a diagnostic is useful. Take, for instance, the task of modeling knowledge including misconcepts. In this case, if a problem x is solvable in a certain competence state κ, that does not necessarily imply that x can be solved when a misconcept is joined to κ; therefore, in this case, any definition of an interpretation function leading to a *monotone* representation function would be inadequate[5].

In many situations, however, special structural assumptions are indicated. The following considerations concentrate on diagnostics with special properties. In the next section, the concept of an *order-stable diagnostic* is introduced, then, in Section 4, the concept of a *union-stable diagnostic* as a special order-stable diagnostic is defined and analyzed more in depth.

3 Order-stable diagnostics

The family $\mathcal{K}$ of competence states of a competence structure $(\mathcal{E}, \mathcal{K})$ can be regarded as being partially ordered through the subset relation $\subseteq$, that is, $(\mathcal{K}, \subseteq)$ is a partially ordered set. Koppen (1989) has shown that a partially ordered family of states can be interpreted in the frame of knowledge modeling in several interesting ways, e.g. as a family of "maximal learning-paths" that generalizes the concept of a Guttman scale. Adopting that interpretation, let us assume for the rest of this paper that the elementary competencies in $\mathcal{E}$ are "proper" knowledge elements and not misconcepts. In the following, we introduce the general concept of an *order-stable diagnostic.*

The concept of an order-stable diagnostic

In general, we call a (finite, nonempty) family $\mathcal{K}$ of competence states ordered by a partial order $\leq$ a *competence ordering* and denote it by $(\mathcal{K}, \leq)$. If a competence structure $(\mathcal{E}, \mathcal{K})$ is ordered by $\subseteq$ we write $(\mathcal{K}, \subseteq)$ for the competence ordering. According to this, a performance structure $(\mathcal{A}, \mathcal{P})$ equipped with the partial order $\subseteq$ is called a *performance ordering* and denoted by $(\mathcal{P}, \subseteq)$.

[4]The problem of "fuzzy competence assessment" is analyzed in another paper.

[5]For instance, Düntsch & Gediga (1995) define the concept of a "skill function" and the notion of "knowledge state" in such a way that the resulting "problem function" is provable as being monotone.

Whenever the level of competence is modeled as a competence ordering $(\mathcal{K}, \leq)$, a natural way of looking for an adequate performance representation suggested by the usual mathematics is to look for a performance ordering $(\mathcal{P}, \subseteq)$ and a surjective order-preserving representation function $p : \mathcal{K} \longrightarrow p(\mathcal{K}) = \mathcal{P}$. This leads to the concept of an *order-stable diagnostic.*

3.1 **Definition:** Let $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ be a diagnostic with the competence ordering $(\mathcal{K}, \leq)$ and the performance ordering $(\mathcal{P}, \subseteq)$. The performance ordering $(\mathcal{P}, \subseteq)$ is called an *order-preserving representation* for $(\mathcal{K}, \leq)$, when the representation function $p : \mathcal{K} \longrightarrow \mathcal{P}$ is monotone (order-preserving), that is, when

$$\kappa \leq \lambda \implies p(\kappa) \subseteq p(\lambda)\,, \quad \textit{for all } \kappa, \lambda \in \mathcal{K}\,.$$

Then the diagnostic $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is called an *order-stable diagnostic.*

We will not develop here fully the theory of order-stable diagnostics. However, the *representation problem for competence orderings* should be formulated and a solution should be derived at this point. The results will be needed for the theory of union-stable diagnostics which we will concentrate on later.

Conditions of order-preserving representation

If $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is a diagnostic with the competence ordering $(\mathcal{K}, \leq)$ and the performance ordering $(\mathcal{P}, \subseteq)$, then the monotony of the representation function can easily be checked. However, the essential aim of the modeling approach followed here, is a *constructive* one: Given a modeled competence ordering, the central task is to construct an order-preserving performance representation. For this, we have to know which formal properties a set of problems must fulfill to guarantee the required representation quality. Thus, we have the following

> **representation problem for competence orderings:**
>
> Given a competence ordering $(\mathcal{K}, \leq)$ and a set $\mathcal{A}$ of problems interpreted in $\mathcal{K}$ through $k(\mathcal{A}) := \{k_x \in \wp(\mathcal{K}) \mid x \in \mathcal{A}\}$.
>
> Which conditions must the set $\mathcal{A}$ resp. the set $k(\mathcal{A})$ satisfy in order to ensure that the induced representation function $p : \mathcal{K} \longrightarrow \wp(\mathcal{A})$ is monotone and thus $(\mathcal{A}, \mathcal{P})$, with $\mathcal{P} := p(\mathcal{K})$, is an order-preserving representation for $(\mathcal{K}, \leq)$?

The following proposition reveals a solution for this representation problem. It formulates two logically equivalent easily checkable criteria for the monotony of the representation function.

3.2 **Proposition**: *Let* $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *be a diagnostic with the competence ordering* $(\mathcal{K}, \leq)$. $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *is an order-stable diagnostic (that is, p is monotone), if and only if for each problem* $x \in \mathcal{A}$ *with concept* k_x *the two logically equivalent conditions are satisfied:*

(i) $\kappa \in k_x \wedge \kappa \leq \lambda \implies \lambda \in k_x$, *for all* $\kappa, \lambda \in \mathcal{K}$;

(ii) $\kappa \in k_x^- \wedge \lambda \leq \kappa \implies \lambda \in k_x^-$, *for all* $\kappa, \lambda \in \mathcal{K}$.

Proof: (i) Let $\kappa, \lambda \in \mathcal{K}$. Then

$$\begin{aligned}
&\kappa \leq \lambda \implies p(\kappa) \subseteq p(\lambda) \\
&\quad \textit{iff } \forall x \in \mathcal{A}[\kappa \leq \lambda \implies (x \in p(\kappa) \implies x \in p(\lambda))] \\
&\quad \textit{iff } \forall x \in \mathcal{A}[\kappa \leq \lambda \implies (\kappa \in k_x \implies \lambda \in k_x)] \\
&\quad \textit{iff } \forall x \in \mathcal{A}[\kappa \in k_x \wedge \kappa \leq \lambda \implies \lambda \in k_x].
\end{aligned}$$

(ii) With $\lambda \notin k_x \iff \lambda \in k_x^-$ the logical equivalence of (i) und (ii) is obvious. □

From the constructional perspective of the competence-performance modeling the criteria given in 3.2 are quite interesting: Conditions for the monotony of the representation function are not *"global"* conditions for the whole set $\mathcal{A}$ of problems; rather they are *"local"* conditions for each single problem. From this follows for the

> **construction of order-stable diagnostics:**
>
> For a given competence ordering the construction of an order-preserving representation can proceed *step by step*. If in an order-stable diagnostic $(\mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ the set $\mathcal{A}$ of problems is extended to the set $\mathcal{A}' := \mathcal{A} \cup \{x\}$ of problems by one problem x satisfying (i, ii) from 3.2, then the resulting diagnostic is order-stable too.

Thus, an order-preserving representation can be constructed step by step becoming an increasingly sharper image of the given competence ordering.

Because the condition for order-stable representations is a *local* condition, further considerations may be directed to the single problem. Let us first introduce a denotation.

3.3 **Definition**: Let $(\mathcal{K}, \leq)$ be a competence ordering. A problem x interpreted in $\mathcal{K}$ through k_x (resp. k_x^-) is called *order-stable* (with respect to $(\mathcal{K}, \leq)$), when the conditions (i) and (ii) from Proposition 3.2 are satisfied.

The concept of an *order-stable problem* may now be characterized as follows: A problem x interpreted in the competence ordering $(\mathcal{K}, \leq)$ is *order-stable* if and only if the following holds: Whenever x is solvable in a certain competence state $\kappa \in \mathcal{K}$, then x is also solvable in each competence state $\lambda \in \mathcal{K}$ "above" κ (that is $\kappa \leq \lambda$). This characterization meets the idea of problems "representative for a competence ordering" precisely:

- Comparing interindividual differences in the solution behavior of subjects on a set of problems, if problem x is solved by a subject in a certain competence state κ, then x should be solved by each subject in a higher competence state $\lambda \geq \kappa$.

- In the context of learning, if a subject proceeds from one competence state κ to a higher state $\lambda \geq \kappa$, then each problem x the subject was capable of solving in κ should also be solvable in λ.

Before proceeding to a more compact characterization of the concept of an order-stable problem, let us first note some aspects of order-stable problems which are needed later. (For a partially ordered set $(P, \preceq)$ we use the denotation Max P resp. Min P for the subset of the maximal resp. minimal elements in P with respect to the partial order $\preceq$.)

3.4 **Lemma:**

(1) *Let* $(\mathcal{K}, \leq)$ *be a competence ordering and* x *a problem that is interpreted in* $\mathcal{K}$ *through* k_x *and order-stable in* $(\mathcal{K}, \leq)$. *Then*

$$\text{Min } \mathcal{K} \not\subseteq k_x \text{ and } (\text{Max } \mathcal{K}) \cap k_x \neq \emptyset \,.$$

(2) *Let* $(\mathcal{K}, \subseteq)$ *be a competence ordering and* x *a problem that is interpreted in* $\mathcal{K}$ *through* k_x *and order-stable in* $(\mathcal{K}, \subseteq)$. *Then*

$$\emptyset \in \mathcal{K} \implies \emptyset \notin k_x ; \quad \mathcal{E} \in \mathcal{K} \implies \mathcal{E} \in k_x \,.$$

Proof:

(1) In case Min $\mathcal{K} \subseteq k_x$, it would follow from the definition of an order-stable problem that $\mathcal{K} \subseteq k_x$. That is in contradiction to 2.2. In case $(\text{Max } \mathcal{K}) \cap k_x = \emptyset$ it would be Max $\mathcal{K} \subseteq k_x^-$, thus $\mathcal{K} \subseteq k_x^-$, thus $k_x = \emptyset$, which contradicts 2.2 as well.

(2) For a competence ordering $(\mathcal{K}, \subseteq)$ with $\emptyset \in \mathcal{K}$ is Min $\mathcal{K} = \{\emptyset\}$. Then from (1) follows $\emptyset \notin k_x$. The second assertion follows the same way. □

Remark: Conversely, the conditions for k_x in 3.4(1) imply $k_x \neq \mathcal{K}$ and $k_x \neq \emptyset$; thus, for an *order-stable* problem these conditions are equivalent to the postulates (k1) and (k2) in Definition 2.2.

From Lemma 3.4, we obtain the following easily verified consequence which we will need later.

3.5 **Consequence:** *In an order-stable diagnostic* $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *the following hold:*

$$\emptyset \in \mathcal{K} \implies p(\emptyset) = \emptyset \in \mathcal{P}; \quad \mathcal{E} \in \mathcal{K} \implies p(\mathcal{E}) = \mathcal{A} \in \mathcal{P}.$$

Characterizations of order-stable problems

Now, we will present a more manageable form of order-stable problems. For this, we introduce some definitions and notations from the theory of partially ordered sets (see e.g. Davey & Priestley, 1990, pp. 13-14).

3.6 **Definition and Notation:**

Let $(P, \preceq)$ be a partially ordered set, $Q \subseteq P$, $z \in P$

Q is called an *order ideal*, when $\forall x, y \in P\, (x \in Q \wedge y \preceq x \implies y \in Q)$.

Q is called an *order filter*, when $\forall x, y \in P\, (x \in Q \wedge x \preceq y \implies y \in Q)$.

Consider the following sets:

$$\uparrow Q := \{y \in P \mid \exists x \in Q\, (x \preceq y)\}; \qquad \uparrow z := \{y \in P \mid z \preceq y\};$$
$$\downarrow Q := \{y \in P \mid \exists x \in Q\, (y \preceq x)\}; \qquad \downarrow z := \{y \in P \mid y \preceq z\}.$$

The set $\uparrow Q$ (resp. $\uparrow z$) is the smallest order filter containing Q (resp. $\{z\}$); Q is an order filter iff $Q = \uparrow Q$; the dual statements hold for $\downarrow Q$ (resp. $\downarrow z$).

For finite P, each order filter $\uparrow Q$ can be written in the form

$$\uparrow Q = \uparrow \text{Min } Q = \bigcup \{\uparrow q \mid q \in \text{Min } Q\},$$

and, accordingly, each order ideal $\downarrow Q$ in P in the form

$$\downarrow Q = \downarrow \text{Max } Q = \bigcup \{\downarrow q \mid q \in \text{Max } Q\}.$$

Using these notions, condition (i) in 3.2 means that k_x is an *order filter*, condition (ii) means that the complementary concept k_x^- is an *order ideal* in $(\mathcal{K}, \leq)$. Thus, we immediately obtain the following characterizations of order-stable problems:

3.7 **Proposition**: *Let* $(\mathcal{K}, \leq)$ *be a competence ordering,* x *a problem with the concept* $k_x \subseteq \mathcal{K}$. *The following are equivalent:*

(1) *Problem* x *is order-stable.*

(2) *There exists a subset* $\mathcal{U} \subseteq \mathcal{K}$ *of competence states so that*

$$k_x = \uparrow\mathcal{U} = \bigcup\{\uparrow\nu_x \mid \nu_x \in \mathcal{U}\}.$$

(3) *There exists a subset* $\mathcal{V} \subseteq \mathcal{K}$ *of competence states so that*

$$k_x^- = \downarrow\mathcal{V} = \bigcup\{\downarrow\mu_x \mid \mu_x \in \mathcal{V}\}.$$

If we replace in the preceding proposition $\mathcal{U}$ resp. $\mathcal{V}$ by the uniquely determined set of minimal resp. maximal elements in k_x using the short denotations $\hat{k}_x :=$ Min k_x resp. $\check{k}_x^- :=$ Max k_x^-, then we arrive at a compact criterion for order-stable problems.

3.8 **Proposition**: *Let* $(\mathcal{K}, \leq)$ *be a competence ordering. A problem* x *with concept* $k_x \subseteq \mathcal{K}$ *is order-stable if and only if*

$$k_x = \uparrow\hat{k}_x = \bigcup\{\uparrow\nu_x \mid \nu_x \in \hat{k}_x\}, \quad \text{or equivalently,} \quad k_x^- = \downarrow\check{k}_x^- = \bigcup\{\downarrow\mu_x \mid \mu_x \in \check{k}_x^-\}.$$

It is very important for practical use, that the concept k_x of an order-stable problem x in a competence ordering $(\mathcal{K}, \leq)$ is completely characterized by the set $\hat{k}_x$ of the minimal competence states, in which x is solvable (and dually by the set $\check{k}_x^-$ of the maximal competence states, in which x is *not* solvable). Occasionally, we refer to $\hat{k}_x$ as the *minimal interpretation* of the order-stable problem x in $(\mathcal{K}, \leq)$. If all the problems of a set $\mathcal{A}$ interpreted in $(\mathcal{K}, \leq)$ by $k : \mathcal{A} \longrightarrow \wp(\mathcal{K})$ are order-stable in $(\mathcal{K}, \leq)$, then k is uniquely determined by the composite function $\hat{k} :=$ Min $\circ\, k$, which assigns to each $x \in \mathcal{A}$ the set $\hat{k}_x :=$ Min k_x. We call $\hat{k}$ the *interpretation (function)* of $\mathcal{A}$ in $(\mathcal{K}, \leq)$ as well as k. This type of interpretation function is introduced in Düntsch & Gediga (1995) under the notion of a "skill function". From the systematical standpoint followed in this paper it should be clear that several implicit assumptions are involved in that notion.

4 Union-stable diagnostics

The concept of a "knowledge space" plays an important role in Doignon & Falmagne's theory because several formally equivalent representations are available (see Section 1). This motivates developing a structurally adequate extension of the notion of a "knowledge space" within the competence-performance approach. In this section we introduce the concept of a *union-stable diagnostic* and analyze some properties of this new concept.

The concept of a union-stable diagnostic

Following the guidelines for transforming the knowledge structures theory into the competence-performance structures approach suggests replacing the concept of a "knowledge space" by two interpretatively different concepts within the competence-performance terminology.

- Let $(\mathcal{E}, \mathcal{K})$ be a competence structure. If $\mathcal{K}$ is stable under union and contains $\emptyset$, then $(\mathcal{E}, \mathcal{K})$ is called a *competence space.* Often $(\mathcal{E}, \mathcal{K})$ is denoted by $(\mathcal{K}, \cup)$.
- Let $(\mathcal{A}, \mathcal{P})$ be a performance structure. If $\mathcal{P}$ ist stable under union and contains $\emptyset$, then $(\mathcal{A}, \mathcal{P})$ is called a *performance space.* The short denotation is $(\mathcal{P}, \cup)$.

4.1 **Remark:** The close connection between the subset relation $\subseteq$ and the union operation $\cup$ via

$$M \subseteq N \iff M \cup N = N, \quad \text{for every two sets } M, N,$$

implies that each competence space $(\mathcal{K}, \cup)$ can be conceived as a competence ordering $(\mathcal{K}, \subseteq)$. The same holds for a performance space $(\mathcal{P}, \cup)$.

With the reinterpretation of a knowledge space as a background, we will specify the introduced concept of a diagnostic in order to utilize the favorable properties of knowledge spaces within the framework of the competence-performance conception. We will define the concept of a *union-stable diagnostic* and legitimize this concept by showing some of its interesting properties for practical applications; further, we will investigate the requirements for a goal-directed construction of a union-stable diagnostic.

Let $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ be a diagnostic with a competence space $(\mathcal{K}, \cup)$. Common mathematical methods suggest postulating that the representation function p should be *union-preserving*, that is, p should map the union of some competence states to the union of the images of these states, that is

$$(4.2) \qquad p(\kappa \cup \lambda) \; = \; p(\kappa) \cup p(\lambda) \,, \quad \textit{for all} \;\; \kappa, \lambda \in \mathcal{K}\,.$$

In order to obtain a consistent concept of a union-stable diagnostic we first prove the following lemma:

4.3 **Lemma:** *Let* $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *be a diagnostic,* $(\mathcal{K}, \cup)$ *a competence space and* $p : \mathcal{K} \longrightarrow \mathcal{P}$ *union-preserving. Then the following holds:*

(1) *With respect to the partial order* $\subseteq$ *the function* p *is especially order-preserving, that is,* $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *is an order-stable diagnostic.*

(2) $\mathcal{P} = p(\mathcal{K})$ *is union-stable;*

(3) $p(\emptyset) = \emptyset \in \mathcal{P}; \quad p(\mathcal{E}) = \mathcal{A} \in \mathcal{P};$

(2) *and* (3) *mean that* $(\mathcal{P}, \cup)$ *is a performance space.*

Proof: (1) That p is monotone follows, using 4.1 and (4.2), from

$$\kappa \subseteq \lambda \implies p(\kappa) \cup p(\lambda) = p(\kappa \cup \lambda) = p(\lambda) \implies p(\kappa) \subseteq p(\lambda)\,.$$

(2) Let X, $Y \in \mathcal{P}$. Since p is surjective, there exist $\kappa, \lambda \in \mathcal{K}$ with $p(\kappa) = X$ and $p(\lambda) = Y$; note that $\kappa \cup \lambda \in \mathcal{K}$. Since p is $\cup$−preserving,

$$X \cup Y \; = \; p(\kappa) \cup p(\lambda) \; = \; p(\kappa \cup \lambda) \in \mathcal{P}.$$

(3) The competence space $(\mathcal{K}, \cup)$ contains $\emptyset$ by definition and $\mathcal{E}$ since $\mathcal{K}$ is union-stable and finite. Now, (1) implies (3) using 3.5. □

Now, we can introduce the following definition.

4.4 **Definition:** Let $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ be a diagnostic, $(\mathcal{K}, \cup)$ a competence space, $(\mathcal{P}, \cup)$ a performance space, and $p : \mathcal{K} \longrightarrow \mathcal{P}$ a union-preserving representation function according to (4.2). Then the representation $(\mathcal{P}, \cup)$ of $(\mathcal{K}, \cup)$ is called a *union-preserving representation* of $(\mathcal{K}, \cup)$; the diagnostic $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is called a *union-stable diagnostic.*

A readily apparent advantage of the concept of a union-stable diagnostic is its property of allowing the definition resp. computation of the representation function to be restricted on the basis of the competence space.

4.5 **Proposition:** *Let* $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ *be a union-stable diagnostic and* $\mathcal{B}(\mathcal{K})$ *the basis of the competence space* $\mathcal{K}$. *If* $\kappa \in \mathcal{K}$ *is a competence state represented as* $\kappa = \bigcup \mathcal{B}_\kappa(\mathcal{K})$ *with an appropriate subset* $\mathcal{B}_\kappa(\mathcal{K}) \subseteq \mathcal{B}(\mathcal{K})$ *of the basis of* $\mathcal{K}$, *then the corresponding performance state* $p(\kappa)$ *is obtained as*

$$p(\kappa) = \bigcup \{p(\beta) \mid \beta \in \mathcal{B}_\kappa(\mathcal{K})\}\,.$$

Since each performance state $Z \in \mathcal{P}$ is the image of some competence state κ (see 2.5(2)), it follows from Proposition 4.5, that each performance state can be written as a union of elements of $p(\mathcal{B}(\mathcal{K}))$; therefore $p(\mathcal{B}(\mathcal{K}))$ must contain the basis $\mathcal{B}(\mathcal{P})$ of the performance space, that is $\mathcal{B}(\mathcal{P}) \subseteq p(\mathcal{B}(\mathcal{K}))$. We will see later which elements of $p(\mathcal{B}(\mathcal{K}))$ belong to the basis $\mathcal{B}(\mathcal{P})$ of the performance space. Knowing this, the basis and therefore the performance space as a whole can be (re-)constructed completely from the basis $\mathcal{B}(\mathcal{K})$ of a competence space in connection with the interpretation of the set of union-stable problems in $\mathcal{B}(\mathcal{K})$.

Due to the union-stable structures both on the set of competence and the set of performance states and particularly due to the union-preserving representation function, the concept of a union-stable diagnostic includes quite a few favorable properties. However, which conditions can ensure that the representation function is union-preserving ?

Conditions of union-preserving representation

Clearly, because the representation function is completely determined by the interpretation function, these conditions will depend on the specific type of problem set. We formulate the problem at hand as the

> **representation problem for competence spaces:**
>
> Given a competence space $(\mathcal{K}, \cup)$.
> Which conditions must a set of problems $\mathcal{A}$ resp. its interpretation $k(\mathcal{A}) := \{k_x \in \wp(\mathcal{K}) \mid x \in \mathcal{A}\}$ in $(\mathcal{K}, \cup)$ satisfy so that the induced representation function $p : \mathcal{K} \longrightarrow \wp(\mathcal{A})$ generates a union-preserving representation $(\mathcal{P}, \cup)$, with $\mathcal{P} := p(\mathcal{K})$ of $(\mathcal{K}, \cup)$; in other words: that $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is a union-stable diagnostic ?

In the following, we will focus our attention on explicating conditions for the constructability of union-stable diagnostics. As will be seen, these conditions can essentially be described as certain formal properties of problems. Further considerations are then directed to various characterizations of problems with these properties (later called "union-stable problems").

The next proposition reveals necessary and sufficient conditions for problems appropriate for the construction of a union-stable diagnostic.

4.6 **Proposition:** *Let $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ be a diagnostic, $(\mathcal{K}, \cup)$ a competence space. The representation function $p : \mathcal{K} \longrightarrow \mathcal{P}$ is union-preserving, hence, $(\mathcal{P}, \cup)$ is a union-preserving representation of $(\mathcal{K}, \cup)$ resp. $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is a union-stable diagnostic, if and only if for each problem $x \in \mathcal{A}$ and all $\kappa, \lambda \in \mathcal{K}$*

(i) $\kappa \in k_x \wedge \kappa \subseteq \lambda \implies \lambda \in k_x$,

(ii) $\kappa \cup \lambda \in k_x \implies \kappa \in k_x \vee \lambda \in k_x$;

or, equivalently,

(i') $\kappa \in k_x^- \wedge \lambda \subseteq \kappa \implies \lambda \in k_x^-$,

(ii') $\kappa \in k_x^- \wedge \lambda \in k_x^- \implies \kappa \cup \lambda \in k_x^-$.

Proof:

($\Rightarrow$) Let p be union-preserving according to (4.2). Then, $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is an order-stable diagnostic according to 4.3(1) and (i) resp. (i') holds for each $x \in \mathcal{A}$ according to 3.2. Now, let $\kappa \cup \lambda \in k_x$, for some $\kappa, \lambda \in \mathcal{K}$ and $x \in \mathcal{A}$, that is, by 2.7, $x \in p(\kappa \cup \lambda)$. Since p is union-preserving, $x \in p(\kappa) \cup p(\lambda)$, hence $x \in p(\kappa) \vee x \in p(\lambda)$, and therefore, by 2.7 again, $\kappa \in k_x \vee \lambda \in k_x$. (ii') is logically equivalent to (ii).

($\Leftarrow$) Given (i), (ii). For $\kappa, \lambda \in \mathcal{K}$ we have only to show $p(\kappa \cup \lambda) = p(\kappa) \cup p(\lambda)$ (according to 4.3 $(\mathcal{P}, \cup)$ is then a performance space). Let $x \in p(\kappa) \cup p(\lambda)$, that is, $x \in p(\kappa) \vee x \in p(\lambda)$; assume $x \in p(\kappa)$, i.e. $\kappa \in k_x$. Since $\kappa \subseteq \kappa \cup \lambda$, it holds, using (i), $\kappa \cup \lambda \in k_x$, i.e. $x \in p(\kappa \cup \lambda)$. Therefore, $p(\kappa) \cup p(\lambda) \subseteq p(\kappa \cup \lambda)$. Conversely, let $x \in p(\kappa \cup \lambda)$, i.e. $\kappa \cup \lambda \in k_x$. From this follows, using (ii), $\kappa \in k_x \vee \lambda \in k_x$, i.e. $x \in p(\kappa) \vee x \in p(\lambda)$, hence $x \in p(\kappa) \cup p(\lambda)$. Therefore $p(\kappa \cup \lambda) \subseteq p(\kappa) \cup p(\lambda)$. □

Proposition 4.6 describes a very important fact: Conditions for a union-stable representation of a competence space are *"local conditions"*. In order to obtain a union-preserving representation function for the competence space, no global property of the set of problems as a whole is required; instead, the necessary and sufficient conditions for the structure-preserving representation prove to be *conditions for each single problem.* This result is of great importance for the

construction of union-stable diagnostics:

The construction of a union-stable diagnostic on a given competence space can be done step by step. If $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is a union-stable diagnostic and the set of problems $\mathcal{A}$ is extended to $\mathcal{A}' := \mathcal{A} \cup \{x\}$ by a problem x satisfying the conditions of Proposition 4.6, then the resulting diagnostic is union-stable as well.

Characterizations of union-stable problems

In the following, we will characterize those problems which assure that the induced representation function is union-preserving. Let us first introduce a denotation for these types of problems.

4.7 **Definition**: Let $(\mathcal{K}, \cup)$ be a competence space. A problem x interpreted in $\mathcal{K}$ through k_x is called *union-stable*, when for all $\kappa, \lambda \in \mathcal{K}$ conditions (i) and (ii) resp. (i') and (ii') from Proposition 4.6 are satisfied.

4.8 **Remark**: Comparing 4.6 and 3.2, it becomes obvious that each union-stable problem x interpreted in a competence space $(\mathcal{K}, \cup)$ can be regarded as a special order-stable problem and hence can be represented as shown in 3.7 and 3.8. This fact is needed later.

Now, a union-stable problem interpreted in a competence space $(\mathcal{K}, \cup)$ can be characterized through the property of order-stable problems described in the preceding section and, additionally, through the following statement:

> If κ, λ are competence states in $\mathcal{K}$, and if the problem x is solvable in the competence state $\kappa \cup \lambda \in \mathcal{K}$, then the problem x is even solvable in *at least one* of these states κ *or* λ.

This characterization proves to be compatible with the idea of a union-preserving representation of the competence space. This idea includes that each union of competence states should correspond to the union of the assigned performance states. Thus, in any learning process that arrives at the union of some competence states, no problem should be solvable that was not solvable in at least one of the previous competence states. Each problem violating this solvability condition would destroy the preserving-property of the representation function and, hence, cannot be used for a structurally adequate representation of the competence space.

Certainly, the practical consequences of the derived results are debatable. For instance, the impression could arise that the conditions required for a union-stable problem might be rather restrictive. This impression can hardly be dispelled unless by a demonstration of the practical applicability of the theory. All in all, the explicated conditions for union-stable problems are exactly those that guarantee a union-preserving representation of a competence space, and the benefits of a union-stable diagnostic have to be balanced against the costs of these restrictive conditions. Perhaps the study reported in Section 5 that demonstrates the modeling of a knowledge domain in the form of a union-stable diagnostic can prevent the stronger objections.

Let us continue analyzing some formal aspects of union-stable problems. Because the central task in constructing a union-preserving representation for a given competence space is the construction or selection of union-stable problems, in the following some specific characterizations of these types of problems for practical use will be explicated. Essentially, we will present two theorems containing formal criteria for the interpretation k_x of a union-stable problem relative to a certain competence space. (In accordance with Section 1, given a competence structure $(\mathcal{E}, \mathcal{K})$, we use the short denotation $\mathcal{K}_\varepsilon$ for the family of all states in $\mathcal{K}$ containing an elementary competence $\varepsilon \in \mathcal{E}$; accordingly, we use the notation $\mathcal{K}_\varepsilon^- := \mathcal{K} \setminus \mathcal{K}_\varepsilon$.)

4.9 **Proposition**: *Let $(\mathcal{K}, \cup)$ be a competence space with the set $\mathcal{E}$ of elementary competencies; further, let $k_x \subseteq \mathcal{K}$ and $k_x^- \equiv \mathcal{K} \setminus k_x \subseteq \mathcal{K}$ be two families of competence states. The following are equivalent:*

(1) *The problem x with interpretation k_x is a union-stable problem.*

(2) *There exists a greatest competence state $\mu_x^- \in k_x^-$ with $\mu_x^- \neq \mathcal{E}$, so that for k_x resp. equivalently for k_x^- holds:*

$$
\begin{aligned}
k_x &= \{\nu \in \mathcal{K} \mid \nu \not\subseteq \mu_x^-\} &= \bigcup\{\mathcal{K}_\delta \mid \delta \in \mathcal{E} \setminus \mu_x^-\}; \\
k_x^- &= \{\mu \in \mathcal{K} \mid \mu \subseteq \mu_x^-\} &= \bigcap\{\mathcal{K}_\varepsilon^- \mid \varepsilon \in \mathcal{E} \setminus \mu_x^-\}.
\end{aligned}
$$

(3) *There exists a nonempty subset $\varphi \subseteq \mathcal{E}$ of elementary competencies, so that for k_x resp. equivalently for k_x^- holds:*

$$
\begin{aligned}
k_x &= \{\nu \in \mathcal{K} \mid \nu \cap \varphi \neq \emptyset\} &= \bigcup\{\mathcal{K}_\delta \mid \delta \in \varphi\}; \\
k_x^- &= \{\mu \in \mathcal{K} \mid \mu \subseteq \mathcal{E} \setminus \varphi\} &= \bigcap\{\mathcal{K}_\varepsilon^- \mid \varepsilon \in \varphi\}.
\end{aligned}
$$

Proof:

(1) $\Rightarrow$ (2). Let x, with $\emptyset \neq k_x \subset \mathcal{K}$ (according to 2.2), be a union-stable problem. Then, according to 4.6(ii'), k_x^- is closed under union, and $\mu_x^- := \bigcup k_x^-$ is with respect to $\subseteq$ the greatest element in k_x^-. According to 3.7, k_x^- can be written as $k_x^- = \{\mu \in \mathcal{K} \mid \mu \subseteq \mu_x^-\} \equiv (\downarrow \mu_x^-)_\subseteq$. The assumption $\mu_x^- = \mathcal{E} \in k_x^-$, would imply $\mathcal{E} \notin k_x$ which is in contradiction to 3.4(2). Further,

$$\begin{aligned} k_x \equiv \mathcal{K} \setminus k_x^- &= \{\nu \in \mathcal{K} \mid \nu \not\subseteq \mu_x^-\} \\ &= \{\nu \in \mathcal{K} \mid \exists \delta \in \nu (\delta \in \mathcal{E} \setminus \mu_x^-)\} = \bigcup \{\mathcal{K}_\delta \mid \delta \in \mathcal{E} \setminus \mu_x^-\}, \end{aligned}$$

and, equivalently,

$$k_x^- \equiv \mathcal{K} \setminus k_x = \mathcal{K} \setminus \bigcup \{\mathcal{K}_\delta \mid \delta \in \mathcal{E} \setminus \mu_x^-\} = \bigcap \{\mathcal{K}_\delta^- \mid \delta \in \mathcal{E} \setminus \mu_x^-\}.$$

(2) $\Rightarrow$ (3). Let $\varphi := \mathcal{E} \setminus \mu_x^-$. Because $\mu_x^- \neq \mathcal{E}$ is $\varphi \neq \emptyset$. Using $\varphi = \mathcal{E} \setminus \mu_x^- \iff \mu_x^- = \mathcal{E} \setminus \varphi$, we obtain for $\mu \in \mathcal{K}$, $\varepsilon \in \mathcal{E}$

$$\begin{aligned} &\mu \subseteq \mu_x^- \iff \mu \subseteq \mathcal{E} \setminus \varphi \quad \text{and} \quad \varepsilon \in \mathcal{E} \setminus \mu_x^- \iff \varepsilon \in \varphi \\ \text{resp.} \quad &\nu \not\subseteq \mu_x^- \iff \nu \cap \varphi \neq \emptyset. \end{aligned}$$

From this the claimed identities follow immediately.

(3) $\Rightarrow$ (1). For $\varphi \subseteq \mathcal{E}$, with $\varphi \neq \emptyset$, let be $k_x^- := \{\mu \in \mathcal{K} \mid \mu \subseteq \mathcal{E} \setminus \varphi\}$. (It is clear that the various representations of k_x resp. k_x^- are equivalent.) It follows that $\mathcal{E} \notin k_x^-$, thus $k_x \neq \emptyset$, and since $\emptyset \in k_x^-$ it is $k_x \neq \mathcal{K}$. Thus, (k1), (k2) from 2.2 are satisfied. Obviously, k_x fulfills the conditions from 4.6/4.7 for a union-stable problem. □

To express 4.9(3) somewhat differently, the statement for a union-stable problem x requires the existence of a nonempty subset $\varphi \subseteq \mathcal{E}$ of elementary competencies (specific for x) so that for all $\nu \in \mathcal{K}$: $\nu \in k_x \iff \exists \varepsilon \in \varphi (\varepsilon \in \nu)$; thus a competence state belongs to k_x if and only if it contains at least one elementary competence of a set $\varphi \subseteq \mathcal{E}$ specific for x.

Remark: The implication (3) $\Rightarrow$ (1) in connection with the preceding results states that if all problems are interpreted in the competence space $(\mathcal{K}, \cup)$ in the described way, then, the resulting performance structure is a performance space. This statement may be regarded as a generalization of Proposition 1.3.1 in Doignon (1994). This is because all subsets of the family S of skills automatically constitute a (special) competence space, and the applied skill assignment (note: not each multiskill assignment!) automatically satisfies the explicated conditions for union-stable problems.

Through 4.9(2) the complementary concept k_x^- of a union-stable problem x is characterized very clearly and handily as an order ideal

$$k_x^- = \{\mu \in \mathcal{K} \mid \mu \subseteq \mu_x^-\} = (\downarrow \mu_x^-)_{\subseteq},$$

with one maximal competence state $\kappa \in \mathcal{K}$. For the problem concept k_x, however, a more convenient description is available. The following theorem provides a characterization for the problem concept k_x of a union-stable problem x using 3.7/3.8 and states from the basis $\mathcal{B}(\mathcal{K})$ or the surmise system $(\mathcal{K},\sigma)$ of the competence space $(\mathcal{K},\cup)$.

4.10 **Proposition:** *Let $(\mathcal{K},\cup)$ be a competence space with the set $\mathcal{E}$ of elementary competencies, the basis $\mathcal{B}(\mathcal{K})$, and the surmise function σ. Further, let $k_x \subseteq \mathcal{K}$ and $k_x^- \equiv \mathcal{K}\setminus k_x \subseteq \mathcal{K}$ be two families of competence states. The following are equivalent:*

(1) *The problem x with interpretation k_x is a union-stable problem.*

(2) *There exists a greatest competence state $\mu_x^- \in k_x^-$ with $\mu_x^- \neq \mathcal{E}$, so that*

$$\begin{aligned} k_x &= \uparrow\{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \not\subseteq \mu_x^-\} &&= \uparrow \mathrm{Min}\ \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \not\subseteq \mu_x^-\}; \\ k_x &= \uparrow\left(\bigcup\{\sigma(\delta) \mid \delta \in \mathcal{E}\setminus \mu_x^-\}\right) &&= \uparrow \mathrm{Min}\ \left(\bigcup\{\sigma(\delta) \mid \delta \in \mathcal{E}\setminus \mu_x^-\}\right). \end{aligned}$$

(3) *There exists a nonempty subset $\varphi \subseteq \mathcal{E}$ of elementary competencies, so that*

$$\begin{aligned} k_x &= \uparrow\{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\} &&= \uparrow \mathrm{Min}\ \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\}; \\ k_x &= \uparrow\left(\bigcup\{\sigma(\delta) \mid \delta \in \varphi\}\right) &&= \uparrow \mathrm{Min}\ \left(\bigcup\{\sigma(\delta) \mid \delta \in \varphi\}\right). \end{aligned}$$

Proof:

(1) $\Rightarrow$ (2). Let x be a union-stable problem interpreted in $\mathcal{K}$ through k_x. By 4.8 x is order-stable, hence $k_x = \uparrow \mathrm{Min}\ k_x$ according to 3.8. By 4.9(2), there exists a $\mu_x^- \neq \mathcal{E}$ with $k_x = \bigcup\{\mathcal{K}_\delta \mid \delta \in \mathcal{E}\setminus\mu_x^-\}$. Then, recalling that for the competence space $(\mathcal{K},\cup)$ the basis $\mathcal{B}(\mathcal{K})$ is constituted by the sets $\hat{\mathcal{K}}_\varepsilon := \mathrm{Min}\ \mathcal{K}_\varepsilon$ via $\mathcal{B}(\mathcal{K}) = \bigcup\{\hat{\mathcal{K}}_\varepsilon \mid \varepsilon \in \mathcal{K}\}$, and moreover that these sets are identical with the clauses $\sigma(\varepsilon)$ (see Section 1), we obtain

$$\begin{aligned} k_x &= \uparrow \mathrm{Min}\ k_x = \uparrow \mathrm{Min}\ \left(\bigcup\{\mathcal{K}_\delta \mid \delta \in \mathcal{E}\setminus\mu_x^-\}\right) \\ &= \uparrow \mathrm{Min}\ \left(\bigcup\{\hat{\mathcal{K}}_\delta \mid \delta \in \mathcal{E}\setminus\mu_x^-\}\right) = \uparrow \mathrm{Min}\ \left(\bigcup\{\sigma(\delta) \mid \delta \in \mathcal{E}\setminus\mu_x^-\}\right) \\ &= \uparrow \mathrm{Min}\ \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap (\mathcal{E}\setminus\mu_x^-) \neq \emptyset\} = \uparrow \mathrm{Min}\ \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \not\subseteq \mu_x^-\}. \end{aligned}$$

The set identities of the form $\uparrow\omega = \uparrow \mathrm{Min}\ \omega$, for $\omega \subseteq \mathcal{K}$, are obvious.

(2) $\Rightarrow$ (3). With $\varphi := \mathcal{E} \setminus \mu_x^-$, the identities of (3) follow from (2).

(3) $\Rightarrow$ (1). Let $\varphi \subseteq \mathcal{E}$, $\varphi \neq \emptyset$. The various representations for k_x are equivalent. We show: x, with $k_x = \uparrow\{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\}$, is union-stable.
First of all, $k_x \neq \emptyset$, and, since $\emptyset \notin k_x$, $k_x \neq \mathcal{K}$. Because x is order-stable, it remains to show for $\kappa, \lambda \in \mathcal{K}$: $\kappa \cup \lambda \in k_x \implies \kappa \in k_x \vee \lambda \in k_x$.
Let $\kappa \cup \lambda \in k_x$; then, there exists $\beta^\circ \in \mathcal{B}(\mathcal{K})$ with $\beta^\circ \subseteq \kappa \cup \lambda \wedge \beta^\circ \cap \varphi \neq \emptyset$. Then, $(\kappa \cup \lambda) \cap \varphi \neq \emptyset$, hence $\kappa \cap \varphi \neq \emptyset \vee \lambda \cap \varphi \neq \emptyset$. Suppose, $\kappa \cap \varphi \neq \emptyset$. In case $\kappa \in \mathcal{B}(\mathcal{K})$, the assertion is true. Let $\kappa \notin \mathcal{B}(\mathcal{K})$. For $\varepsilon \in \kappa \cap \varphi$ there exists $\beta_\varepsilon \in \sigma(\varepsilon)$ with $\varepsilon \in \beta_\varepsilon \subseteq \kappa$, hence with $\beta_\varepsilon \cap \varphi \neq \emptyset$. Thus, $\kappa \in k_x$. □

The various forms for the interpretation of a union-stable problem relative to competence space suggest the following notions:

4.11 **Definition:** Let $(\mathcal{K}, \cup)$ be a competence space on the set $\mathcal{E}$ of elementary competencies; let x be a union-stable problem interpreted by k_x in $\mathcal{K}$.

(1) The unique subset $\hat{k}_x := \text{Min } k_x$ is called the *minimal interpretation* of x (resp. of k_x).

(2) A nonempty subset $\varphi \subseteq \mathcal{E}$ of elementary competencies with the property

$$k_x = \uparrow\{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\} \equiv \uparrow\text{Min } \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\}$$

is called a *generating set* of k_x.

(3) If φ is a generating set of k_x, then we call
$\tilde{k}_x := \{\beta \in \mathcal{B}(\mathcal{K}) \mid \beta \cap \varphi \neq \emptyset\}$ the *basis interpretation* of x (resp. of k_x),
$k_x^\sigma := \bigcup\{\sigma(\delta) \mid \delta \in \varphi\}$ the *surmise interpretation* of x (resp. of k_x).

The defined notions can apply as criteria for union-stable problems relative to a competence space $\mathcal{K}$: Under the precondition that the concept k_x of a problem x is accepted as being an order filter in $\mathcal{K}$, the problem x is union-stable in $\mathcal{K}$ if and only if x can be interpreted in $\mathcal{K}$ by one (and then every) of the described concepts.

Let us state here, that for the minimal, the basis and the surmise interpretation of a union-stable problem x interpreted in $\mathcal{K}$ by k_x the following holds:
$\hat{k}_x \subseteq k_x^\sigma \subseteq \tilde{k}_x$, with $\hat{k}_x = \text{Min } k_x^\sigma = \text{Min } \tilde{k}_x$.

Preserving-properties in a union-stable diagnostic

Let $(\mathcal{K}, \cup)$ be a competence space and $\mathcal{A}$ a problem set interpreted in $\mathcal{K}$ through a function k; if the problems of $\mathcal{A}$ are union-stable, then the resulting diagnostic

$(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is union-stable. Now the question arises how this diagnostic can be established economically. In the following, we show how the basis $\mathcal{B}(\mathcal{P})$ and the surmise-function s of the representing performance space $(\mathcal{P}, \cup)$ can be constructed knowing only the basis $\mathcal{B}(\mathcal{K})$ of the competence space and the basis interpretation $\tilde{k}_x$ for each problem $x \in \mathcal{A}$.

Proposition 4.5 included that in a union-stable diagnostic $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ the basis $\mathcal{B}(\mathcal{P})$ of the performance space is a subset of $p(\mathcal{B}(\mathcal{K}))$. As was noted earlier (see Section 1), $\mathcal{B}(\mathcal{P})$ is obtained through $\mathcal{B}(\mathcal{P}) = \bigcup\{\hat{\mathcal{P}}_x \mid x \in \mathcal{A}\}$, where for all $x \in \mathcal{A}$ the family $\mathcal{P}_x$ includes all performance states containing x and $\hat{\mathcal{P}}_x := \text{Min } \mathcal{P}_x$ is the set of the minimal elements in $\mathcal{P}_x$. The question to answer now is, how to select the families $\hat{\mathcal{P}}_x$ from $p(\mathcal{B}(\mathcal{K}))$.

Obviously, in each diagnostic $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ the set $\mathcal{P}_x$ $(x \in \mathcal{A})$ can be constructed as the image $p(k_x)$ of the interpretation of x in $\mathcal{K}$ under p, that is $\mathcal{P}_x = p(k_x)$. If $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ is union-stable, for the basis interpretation $\tilde{k}_x$ of a (union-stable) problem $x \in \mathcal{A}$ (using the fact that p is order-preserving according to 4.3) it is easily shown, that $\hat{\mathcal{P}}_x \subseteq p(\tilde{k}_x) \subseteq p(k_x) = \mathcal{P}_x$, thus $\hat{\mathcal{P}}_x = \text{Min } p(\tilde{k}_x)$. On the other hand, for $\tilde{k}_x \subseteq \mathcal{B}(\mathcal{K})$ we obviously have $\text{Min } p(\tilde{k}_x) \subseteq p(\tilde{k}_x) \subseteq p(\mathcal{B}(\mathcal{K}))$.

We summarize these considerations by the following proposition.

4.12 **Proposition**: *For a union-stable diagnostic $(\mathcal{E}, \mathcal{K}, \mathcal{A}, \mathcal{P}, k, p)$ the following statements hold:*

(1) *The basis $\mathcal{B}(\mathcal{P})$ of the performance space $(\mathcal{P}, \cup)$, which is obtained through $\mathcal{B}(\mathcal{P}) = \bigcup\{\hat{\mathcal{P}}_x \mid x \in \mathcal{A}\}$, is a subset of $p(\mathcal{B}(\mathcal{K}))$, that is $\mathcal{B}(\mathcal{P}) \subseteq p(\mathcal{B}(\mathcal{K}))$.*

(2) *The families $\hat{\mathcal{P}}_x$, that constitute the basis $\mathcal{B}(\mathcal{P})$ of the performance space, can be obtained by taking the minimal elements from the image of the basis interpretation $\tilde{k}_x$ of x, that is $\hat{\mathcal{P}}_x = \text{Min } p(\tilde{k}_x)$.*

Taking into account once more that by Doignon & Falmagne's theorem the surmise function s for a performance space $(\mathcal{P}, \cup)$ is connected with the basis $\mathcal{B}(\mathcal{P})$ of $(\mathcal{P}, \cup)$ via $s(x) = \hat{\mathcal{P}}_x$ for each $x \in \mathcal{A}$, we see, that *knowing the basis of the competence space and the interpretation of the union-stable problems on the basis of the competence space is sufficient for constructing the basis, the surmise-function and thus the performance space itself.* The consequences of this finding are of considerable practical use when a union-stable diagnostic has to be established.

5 Empirical application

To test the applicability and utility of the competence-performance conception, several empirical investigations were conducted (Korossy, 1993). In this section we present a reanalysis of an empirical study reported by Korossy (1996)[6]. The knowledge modeling developed and tested in that study is based on the relatively weak concepts of a competence structure and a representing performance structure. For the purpose of illustrating the appropriateness of the above introduced modeling approach for practical application, we adopt from Korossy's study the slightly revised competence structure as a competence space $(\mathcal{K}, \cup)$, and select from the family of problems exactly those problems that prove to be *union-stable* in $(\mathcal{K}, \cup)$ so as to obtain a union-preserving performance space $(\mathcal{P}, \cup)$. The resulting union-stable diagnostic, then, is examined for aspects of validity on the basis of the collected data.

Because the principles of modeling and details of the study are reported in Korossy (1996), the following presentation is restricted to the more formal aspects specific to the knowledge modeling in the framework of a *union-stable diagnostic*. First, however, we briefly describe the knowledge domain selected for the study and the goals of the domain-specific modeling.

Aim: Modeling and validating a network of learning goals

As a specific knowledge domain the area "geometry of right triangles" was selected. This area is a standard section in nearly every curriculum of teaching geometry. The background of the study is a German high school mathematics curriculum (*"Bildungsplan für das Gymnasium"*, Ministerium für Kultus und Sport, 1984), that sets as teaching/learning goals the following topics in the selected knowledge domain:

- Knowledge of the three *Pythagorean Theorems*, in German called "Satz des Pythagoras", "Kathetensatz", and "Höhensatz", that refer to the three theorems P, K, and H listed in Table 1;
- as applications of these theorems: *transformations of areas* (e.g. constructing a square with the same area as a given rectangle) and *calculations of lengths* (e.g. the altitude in an equilateral triangle).

[6]The study reported in this article was supported by the Deutsche Forschungsgemeinschaft under Grant Lu 385/1.

Table 1: The three Pythagorean Theorems P, K, H

Theorem P (Pythagoras' Theorem): *In each right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.* **Theorem K:** *In each right triangle, the square of each leg is the product of the hypotenuse and the leg's projection on the hypotenuse.* **Theorem H:** *In each right triangle, the square of the altitude to the hypotenuse is equal to the product of the segments into which the altitude divides the hypotenuse.*

For these topics a small network of teaching/learning goals is to be reconstructed as a competence space. Table 2 describes the teaching/learning goals to be included with the respective abbreviation and meaning.

Table 2: Set $\{P, K, H, A, Z, T\}$ of teaching/learning goals

Abbreviation	Domain-specific meaning
P	knowledge of the *Satz des Pythagoras*
K	knowledge of the *Kathetensatz*
H	knowledge of the *Höhensatz*
A	knowledge about calculating the *area of a right-angled triangle*
Z	knowledge of *constructing a square with the same area as a given rectangle*
T	knowlege of *properties of tangents on circles*

The three teaching/learning goals P, K, H correspond to the topics *Satz des Pythagoras, Kathetensatz, Höhensatz* listed in the Bildungsplan (see Table 1); the three remaining goals may be understood as specifying the required applications: A as an application of the theorems above in the context of calculating the area of a (rectangled) triangle; Z as an application in the context of transforming a rectangle into a square with the same area as the rectangle; T as an application in the context of calculations with circles, whereby properties of tangents on circles are needed.

With respect to these preconditions of the Bildungsplan and the teaching/learning goals defined in Table 2, the specific goals of knowledge modeling to be presented in this section can now be explicated in the following way:

1. A curricular network of teaching/learning objectives is to be modeled as a *competence space*;

2. suitable problems (i.e. problems with *union-stable* interpretations in the modeled competence space) are to be selected and the resulting performance space is to be designed;

3. the empirically observable solution patterns are to be designated and compared with the hypothetically expected solution patterns (the performance states), and by way of this comparison aspects of the empirical validity of the modeling are to be collected.

In the following two subsections the results of the modeling process (rather than the modeling process itself) are described.

Modeling the competence space

The teaching/learning goals defined in Table 2 were conceived in our study as elementary competencies. With respect to this set $\mathcal{E} = \{P, K, H, A, Z, T\}$ of elementary competencies the following competence space $(\mathcal{K}, \cup)$, with basis $\mathcal{B}(\mathcal{K})$ and surmise function $\sigma : \mathcal{E} \longrightarrow \wp(\wp(\mathcal{E}))$ is defined:

$$\mathcal{B}(\mathcal{K}) = \{K, H, PK, PH, KA, HA, KZ, HZ, PKTA, KHTA\}$$ [7]

$\sigma : \mathcal{E} \longrightarrow \wp(\wp(\mathcal{E}))$

$\varepsilon \in \mathcal{E}$	P	K	H	A	Z	T
$\sigma(\varepsilon)$	$\{PK, PH\}$	$\{K\}$	$\{H\}$	$\{KA, HA\}$	$\{KZ, HZ\}$	$\{PKTA, KHTA\}$

The competence space $(\mathcal{K}, \cup)$ contains 31 competence states.

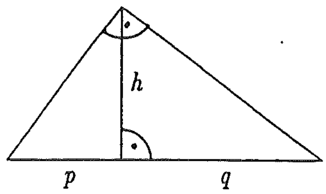
[7] For a shorter notation we write the competence states as sequences of letters instead of as sets. In full, for $\mathcal{B}(\mathcal{K})$ we would write:

$$\mathcal{B}(\mathcal{K}) = \{\{K\}, \{H\}, \{P, K\}, \{P, H\}, \{K, A\}, \ldots, \{P, K, T, A\}, \{K, H, T, A\}\}.$$

Constructing a representing performance structure

As a first step in establishing a performance representation of $(\mathcal{K}, \cup)$ from a set of 10 geometry problems involved in the original study, the set $\mathcal{A} := \{a, b, c, d, e\}$ of the five problems presented in Figure 1 is selected.

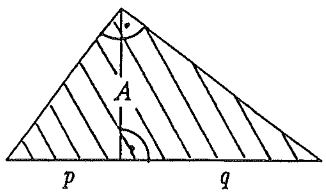
Problem *a*



Given: $q = 7$ cm, $h = 4$ cm

Calculate: p

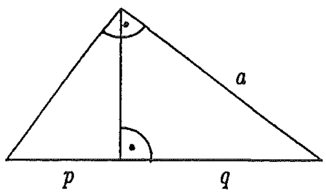
Problem *b*



Given: $p = 4$ cm, $q = 7$ cm

Calculate: area A

Problem *c*



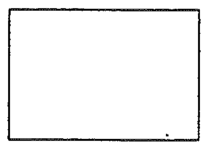
Given: $a = 8$ cm, $q = 6$ cm

Calculate: p

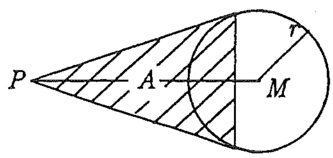
Problem *d*

Given: a rectangle

Construct a square with the same area as the rectangle.



Problem *e*



Given: $\overline{PM} = 6$ cm, $r = 4$ cm

Calculate: area A

Figure 1: The five geometry problems in $\mathcal{A}$.

Based on a solution analysis (see Appendix A.1) this set of problems is interpreted in $\mathcal{K}$ by an interpretation function $k : \mathcal{A} \longrightarrow \wp(\mathcal{K})$. Table 3 shows for each problem $x \in \mathcal{A}$ a generating set $\varphi_x \subseteq \mathcal{E}$ for k_x, the *basis interpretation* $\tilde{k}_x$ and for the purpose of illustration additionally the *surmise interpretation* k_x^σ and the *minimal interpretation* $\hat{k}_x$ of x, whereby it is presupposed that $k_x =\uparrow \tilde{k}_x \equiv\uparrow k_x^\sigma \equiv\uparrow \hat{k}_x$. According to Proposition 4.10 each problem $x \in \mathcal{A}$ is union-stable. In fact, from the original set of 10 problems exactly those are selected that can be interpreted in $(\mathcal{K}, \cup)$ as union-stable problems.

Table 3: Minimal, surmise, basis interpretation for the problems and the representation function $p : \mathcal{B}(\mathcal{K}) \longrightarrow \wp(\mathcal{A})$

$x \in A$	φ_x	K	H	PK	PH	KA	HA	KZ	HZ	$PKTA$	$KHTA$
a	$\{H, P\}$		⊛	⊛	∗		+		+	+	+
b	$\{A\}$					⊛	⊛			+	+
c	$\{K, P\}$	⊛		∗	⊛	+		+		+	+
d	$\{Z\}$							⊛	⊛		
e	$\{T\}$									⊛	⊛
$p(\kappa)$		c	a	ac	ac	bc	ab	cd	ad	$abce$	$abce$

φ_x generating set for the problem concept k_x
$+$ basis interpretation $\tilde{k}_x$ of problem $x \in \mathcal{A}$
$\times$ surmise interpretation k_x^σ of x
$\odot$ minimal interpretation $\hat{k}_x$ of x.

The defined interpretation function k uniquely determines the representation function $p : \mathcal{K} \longrightarrow \wp(\mathcal{A})$, which is union-preserving as a consequence of the union-stable problems. According to 4.3 the performance structure $(\mathcal{A}, \mathcal{P})$, with $\mathcal{P} := p(\mathcal{K})$, is a performance space $(\mathcal{P}, \cup)$. By 4.5, p can be restricted to the basis $\mathcal{B}(\mathcal{K})$ of the competence space $(\mathcal{K}, \cup)$. This restricted representation function $p : \mathcal{B}(\mathcal{K}) \longrightarrow \wp(\mathcal{A})$ can also be found in Table 3.

In accordance with Proposition 4.12, the basis $\mathcal{B}(\mathcal{P})$ of $(\mathcal{P}, \cup)$ can be found within the images of $\mathcal{B}(\mathcal{K})$ under the representation function p: For each $x \in \mathcal{A}$ select from the image set $p(\mathcal{B}(\mathcal{K}))$ the family $p(\tilde{k}_x) \subseteq p(\mathcal{B}(\mathcal{K}))$ and take the minimal states of

$p(\tilde{k}_x)$. By 4.12, $\hat{\mathcal{P}}_x = \text{Min}\ p(\tilde{k}_x)$. Table 4 shows for each $x \in \mathcal{A}$ the families $p(\tilde{k}_x)$ and $\hat{\mathcal{P}}_x \equiv \text{Min}\ p(\tilde{k}_x)$[8].

The basis $\mathcal{B}(\mathcal{P})$ of the performance space is immediately obtained through $\mathcal{B}(\mathcal{P}) = \bigcup\{\hat{\mathcal{P}}_x \mid x \in \mathcal{A}\}$ from Table 4:

$$\mathcal{B}(\mathcal{P}) = \{a, c, ab, ad, bc, cd, abce\}.$$

Moreover, with the families $\hat{\mathcal{P}}_x$, shown in Table 4 for all $x \in \mathcal{A}$, we have generated the surmise-function s for the performance space $(\mathcal{P}, \cup)$; recall that s is related to the families $\hat{\mathcal{P}}_x$ through $s(x) = \hat{\mathcal{P}}_x$ for each $x \in \mathcal{A}$ according to Doignon & Falmagne's theorem (see Section 1).

Table 4: Construction of the families $\hat{\mathcal{P}}_x \equiv s(x) = \text{Min}\ p(\tilde{k}_x)$ using the basis interpretations $\tilde{k}_x$ of the problems $x \in \mathcal{A}$

$x \in \mathcal{A}$	$p(\tilde{k}_x)$	$\hat{\mathcal{P}}_x \equiv s(x) = \text{Min}\ p(\tilde{k}_x)$
a	$\{a, ac, ab, ad, abce\}$	$\{a\}$
b	$\{bc, ab, abce\}$	$\{ab, bc\}$
c	$\{c, ac, bc, cd, abce\}$	$\{c\}$
d	$\{cd, ad\}$	$\{ad, cd\}$
e	$\{abce\}$	$\{abce\}$

The performance space $(\mathcal{P}, \cup)$ contains 15 performance states. It is depicted in Figure 2 as a Hasse diagram (with respect to $\subseteq$). Under each performance state the corresponding class of competence states is placed. Note that Figure 2 illustrates the grain size of the performance representation for the given competence space and thus forecasts the accuracy of individual competence diagnosis in the case that the established modeling is accepted as being valid.

The hypothesis for the reanalysis of the experimental data is, that the observed solution patterns on the problems should agree with the theoretically expected solution patterns, i.e. with the performance states of $\mathcal{P}$.

[8]It should be remarked at this point that a more convenient procedure for obtaining $\hat{\mathcal{P}}_x$ resp. $s(x)$ is to take first the minima $\hat{k}_x \equiv \text{Min}\ (\tilde{k}_x) \equiv \text{Min}\ (k_x^\sigma)$ of the problem interpretations and then Min $p(\hat{k}_x)$. This procedure can easily be verified by looking at Tables 3 and 4.

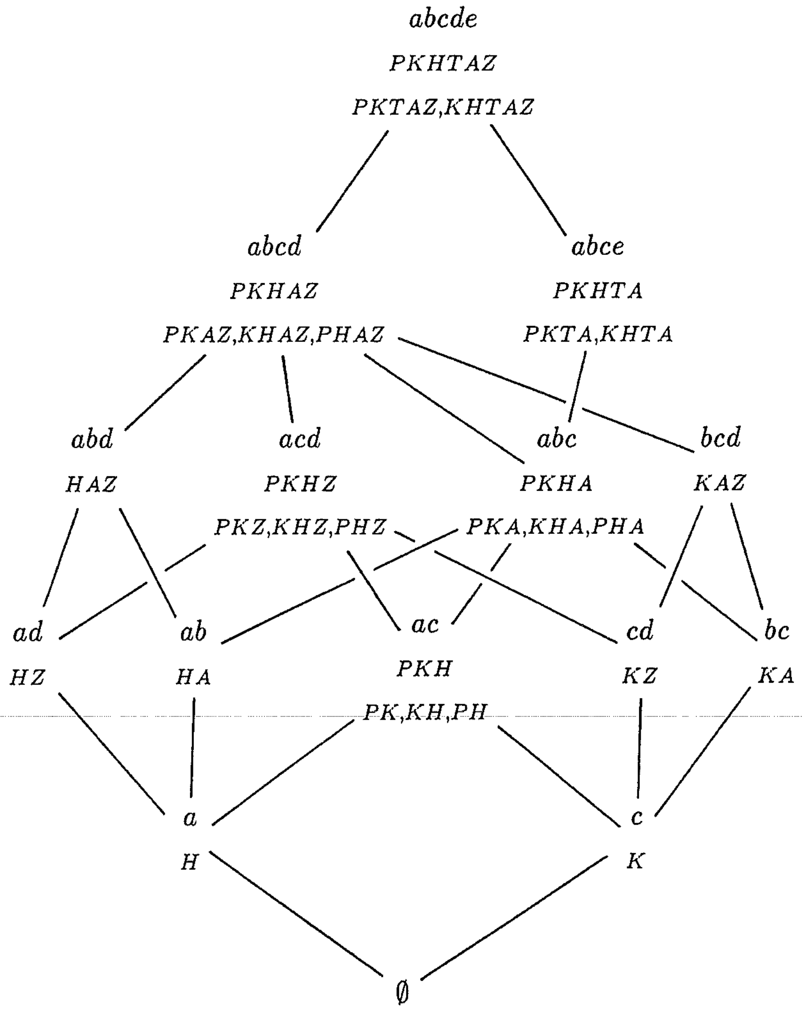


Figure 2: Performance space $(\mathcal{P}, \cup)$ as a Hasse diagram relative to $\subseteq$; under each performance state the class of corresponding competence states is depicted.

Method

Problems

Korossy (1996) included in his performance representation 10 geometry problems[9]. For the reanalysis presented here, from the original set of 10 problems those five problems (see Figure 1) that are union-stable in $(\mathcal{K}, \cup)$ were selected in order to construct a union-preserving performance representation of $(\mathcal{K}, \cup)$.

[9] In the original study the subjects were tested in groups of 6 to 8 persons. In order to minimize the "knowledge transfer" within the test groups, the neighbors were presented different but parallel versions of each problem. Since no obvious differences between the two parallel groups were observed, the data were pooled for evaluation.

Same-type geometry problems as the 10 problems applied in the original study for the representing performance structure of the competence space $(\mathcal{K}, \cup)$ are likely to be found in every textbook on elementary geometry. Most of those problems can be solved using different approaches. Nevertheless, according to the curriculum-oriented competence modeling only solution ways using knowledge from the area of the *Pythagorean theorems* were taken into consideration for the interpretation of the problems; other solution approaches (e.g. applying properties of similarity between rectangled triangles or trigonometrical functions) are excluded.

All 10 problems (and thus the five problems considered here) were ordered within the original test in such a way that for each pair of problems comparable with respect to their (theoretically hypothesized) difficulty *the more difficult problem did not precede the less difficult problem*[10]. This restriction facilitates learning effects that may generate solution patterns incompatible with the theory, but it should prevent demotivating effects than can occur when the first problems are too difficult. The solution paths taken into consideration are presented in Appendix A.1.

For each problem the available solution time was limited to 10 minutes; moreover, a minimum solution time of at least 3 minutes was fixed for each problem in order to prevent careless errors and to provide a maximal chance for actualizing existing competencies.

Subjects

The experiment was conducted with 62 subjects (36 male, 26 female) ranging in age from 15 to 57. Ten persons were excluded from the evaluation because they applied solution methods not included in the competence modeling. There remained 52 subjects (31 male, 21 female) for the evaluation.

Procedure

The experiment was announced as an "investigation on solving geometry problems". It was carried out as a series of group tests each with 6 to 8 persons.

First the subjects were given written instructions for the experimental procedure. They were asked to solve the problems in the given sequence, to stay within the prescribed solution time for each problem, and to avoid careless errors. After distributing the tests each student worked on the problems according to his/her individual speed.

[10]Problem y was defined as *"more difficult than"* a problem x, when x occurs in each clause for y, but not conversely. Obviously this relation allows consistent sequencings of the problems.

Results

While the theory presupposes only two-categorial solution behavior (correct/incorrect), the experimental method (paper and pencil) provided additional information on the applied solution approaches and solution ways. This information was registered and evaluated with regard to the possibility of validating the solution analyses of the problems. Table 6 in Appendix A.2 shows for each person the observed solution way for each problem $x \in \mathcal{A}$ and the solution pattern for the entire set $\mathcal{A}$ of the five problems.

The solutions were evaluated under *two different aspects:*

Evaluation 1: The solution of the problem is entirely correct.

Evaluation 2: The solution way of the problem is correct, but the numerical result is incorrect (e.g. as a result of a computational error).

An overview of the empirically observed solution patterns with respect to the two ways of evaluation is provided by Table 5.

Table 5: Frequency of the performance states

Performance state	Evaluation 1 solution way correct num. result correct	Evaluation 2 solution way correct num. result incorrect
$\emptyset$	9	9
a	5	4
c	1	—
ad	—	—
ab	4	3
ac	1	1
cd	—	—
bc	—	1
abd	—	—
acd	—	—
abc	12	12
bcd	—	—
abcd	3	2
abce	7	10
abcde	6	8
not predicted	4	2
total:	52	52

As Düntsch & Gediga (1995) emphasized there are some serious problems of evaluating the congruence between the expected and the observed solution patterns. The data evaluation in Korossy (1996) is also confronted with these problems. There, two possible ways of a qualitative analysis of the model violating solution patterns are followed and discussed. In the present case, however, we may be allowed to set these problems aside for two reasons: First, our purpose is only to illustrate the principles of an application of the theory developed in the preceding sections; second, the results shown in Tables 5 and 6 appear to have such a high face validity that a statistical evaluation does not seem to be necessary at this point.

As can be seen from Table 5, for all but four subjects the solution patterns following evaluation 1 are in accordance with the expected performance states. For evaluation 2 only two non-expected solution patterns were observed. Moreover, 9 of the 15 performance states were observed following evaluation 1 as well as 9 performance states under evaluation 2.

Discussion

The reanalysis of an empirical study reported in Korossy (1996) was presented primarily for the purpose of illustrating the theoretical concept of a union-stable diagnostic introduced as an extension of Doignon & Falmagne's concept of a knowledge space. The knowledge domain of the Pythagorean theorems was to be designed in the form of a competence space and a union-preserving empirical representation in the form of a performance space. For this, we used a slight modification of the competence structure in the original modeling and selected five problems of the original study in order to get a union-stable diagnostic.

In principle, the original experiment can be regarded as a validation study for the modified competence modeling as well. Whereas the results of the original study based on the whole set of ten problems were to be considered critically in view of the validity of the established modeling (see Korossy, 1996), for the modified model analyzed above the high congruency of the empirically observed and the theoretically expected solution patterns could be taken as an indication for the psychological validity of the underlying competence model and the task analyses. Of course, if there should be further interest in the modeled competence space, the validation process would have to be continued by establishing another performance representation and testing the performance states against the observed solution patterns.

Whenever the competence space is accepted as being psychologically valid, then, in connection with a suitable performance representation, it can be used as a diagnostical framework for an economic qualitative competence diagnosis and goal-oriented adaptive learning/teaching processes. Moreover, because of the union-stable structures on the competence and on the performance space and the union-preserving representation function, much of the practical work can be done in a flexible, convenient, and ecomical way by making use of the equivalent concepts of the (competence/performance) space, the basis of the space, and the according surmise system.

6 Summary and general discussion

In marked contrast to most of the contemporary systems for knowledge representation, in Doignon & Falmagne's *knowledge structures theory* knowledge representation and knowledge diagnosis are closely related. However, the knowledge structures theory suffers from the serious deficiency that the core of the theory is built on purely behavioral concepts that provide no reference to existing theories on domain-specific knowledge-related behavior. Also the approaches of Falmagne et al. (1990), Doignon (1994), and Düntsch & Gediga (1995) to supply knowledge structures with underlying sets of abstract skills seem to be unsatisfactory because they merely generate recodings of empirical data in other structures.

The knowledge structures approach at the present state of development ignores that there exist numerous specific theoretical models for observable behavior in solving certain types of problems that seem appropriate for use in knowledge modeling within the structural approach. With the *competence-performance conception* for the knowledge structures theory outlined in this paper, an attempt has been made to develop a framework for including available domain-specific theories for the task of knowledge modeling. The central idea is that appropriate specific theories should be utilized for the modeling of genuine interpretatively meaningful "skills", capabilities or "competencies" underlying the observable behavior.

The *competence-performance approach* to the theory of knowledge structures assumes (see Section 2) that domain-specific knowledge can be modeled upon two related levels: on the performance level, as a family of certain subsets of a domain-specific set of problems, and, on a non-observational competence level, as a family of *competence states.* Each person is assumed to be (at one time) in exactly one competence state that enables solving certain problems and not others. The family of competence states is presupposed to be established on the basis of domain-specific theorizing. The subset of solvable problems is called the *performance state* of this person. The relationship of the two modeling levels is constructed in a deterministic way: Each problem is mapped by an *interpretation function* to that subset of competence states each of which enables solving the problem (this accounts for alternative solution ways of the problem); this assignment, conversely, determines a *representation function* that maps each competence state to the subset of problems solvable in that state (the representing performance state). The family of competence states, the set of problems, the family of performance states, the interpretation and the representation function together constitute the concept of a *diagnostic* that is the basic concept of the competence-performance approach.

A diagnostic can be structurally enriched by assuming that the family of competence states as well as the family of performance states are structured by order relations or algebraic operations and that the representation function is a structure-preserving morphism. For example, when the families of the competence states and the performance states are constituted as partially ordered sets, and when the representation function is order-preserving, then we get the concept of an *order-stable diagnostic*. In Section 3 diagnostics of this type are briefly described and analyzed.

Doignon & Falmagne's *theory of knowledge spaces* is integrated into the competence-performance approach by replacing the concept of a *knowledge space* by the two concepts of a *competence space* and a *performance space* and requiring the representation function to be union-preserving. This approach constitutes the concept of a *union-stable diagnostic* (see Section 4). The advantageous property of a union-stable diagnostic is that, on the competence level as well as on the performance level, several equivalent concepts for the task of modeling or storage are available (according to Doignon & Falmagne's theory) and that the union-preserving representation function guarantees the structural correspondence of these various modeling concepts on the competence level to those of the performance level. Analysis of the conditions for establishing a union-stable diagnostic reveals that, given a competence space, a representing union-preserving performance space can be constructed step by step by including domain-specific problems that satisfy certain structural demands. The central concepts and results are illustrated by (the reanalysis of) a study that is reported in Section 5.

Some advantages of the competence-performance conception for the knowledge structures theory seem obvious:

(1) Whenever a competence model based on a domain-specific theory (in our case a curriculum in elementary geometry) has been established, a theory-guided step-by-step construction of a structure-preserving performance representation can be tackled. Due to the explicitly defined relationship between competence and performance a domain-specific knowledge modeling can immediately become the object of validation studies by comparing the competence-based performance states with the empirically observed solution patterns on the selected problem set. Performance states incorporate theoretically founded hypotheses on the expected solution patterns; violations of hypotheses can be theoretically analyzed (both in contrast to the behavioral versions of the knowledge structures theory).

(2) Whenever a domain-specific theory is utilized for establishing the competence model, this theory is, by way of validating the competence model, given a chance for empirical validation. When, for example, a certain curriculum theory that suggests a definite sequencing of instructional objectives is utilized for a competence modeling, then any successful empirical validation of that competence model provides empirical support for the curriculum theory as well. In this sense, the competence-performance conception of knowledge modeling may be regarded as an opportunity for empirical theory-validating.

(3) Whenever a competence modeling has been accepted as being psychologically valid, then, in connection with a suitable performance representation, it can be used as a diagnostical framework for qualitative competence diagnosis and goal-oriented adaptive learning/teaching processes.

Nevertheless, several restrictions of the competence-performance conception can hardly be overlooked. Perhaps one of the stronger objections to the proposed approach is that the tools for knowledge modeling available at this developmental state of the theory seem rather restricted. For example, compared with the demands for a subtle knowledge modeling required by some cognitive theories (apart from the difficulties for evaluation mentioned above), the concept of a *state* is a rather weak concept; also, the *interpretation* of a problem within a competence structure seems to be too inflexible in several aspects. In this respect, the modeling approach should be considerably enriched in order to meet the requirements of detailed knowledge diagnosing.

On the other hand, it should be clear that the modeling of competence-performance structures should be conducted in reference to a domain-specific theory; it cannot take the place of domain-specific theorizing. The examples described in Section 1, and the study reported in Section 5 may clarify this point: The applied theory has to reveal the specific characteristics of the to-be-modeled individual knowledge; it must determine which (kind of) states of individual knowledge are to be taken into consideration, and it should guide the selection of problems suitable for an adequate empirical representation. Then, the competence-performance approach provides an apparatus for constructing an explicit relation between competence and performance, that is, for relating the applied theory to the level of empirical observation. In this respect, the usefulness of the competence-performance conception has to be proven by the successful cooperation with domain-specific research.

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A Appendix

A.1 Solution ways for the problems

In the following the solution ways for problems a – e of Figure 1 are outlined. For each problem the simplest alternative solution ways using the Pythagorean theorems are presented; solution ways involving other mathematical methods (for instance similarity of rectangled triangles or trigonometrical functions) are excluded with respect to the curriculum selected for our study.

The definitions of the elements P, K, H, T, A, Z used as elementary competencies are found in Table 2.

Problem a: $\{PK, H\}$

(H)	$h^2 = pq \implies p = \dfrac{h^2}{q}$
(PK)	$a^2 = h^2 + q^2 \wedge a^2 = cq \implies c = \dfrac{h^2+q^2}{q} \implies p = \dfrac{h^2+q^2}{q} - q = \dfrac{h^2}{q}$
	Result: $p \approx 2.29\,cm$

Problem b: $\{KA, HA\}$

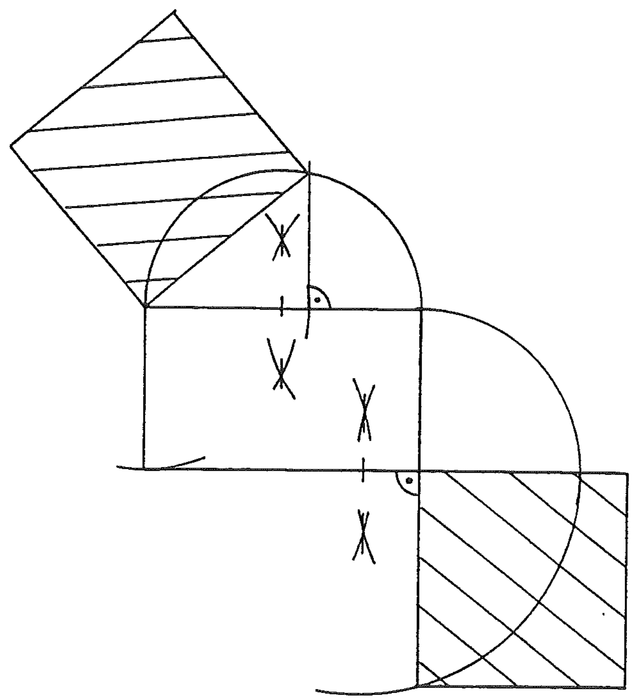
(H)	$h^2 = pq \implies h = \sqrt{pq}$
(A)	$A = \frac{1}{2}ch = \frac{1}{2}(p+q)\sqrt{pq}$
(K)	$a^2 = cq = (p+q)q \implies a = \sqrt{(p+q)q}$
	$b^2 = cp = (p+q)p \implies b = \sqrt{(p+q)p}$
(A)	$A = \frac{1}{2}ab = \frac{1}{2}(p+q)\sqrt{pq}$
(H)	$h^2 = pq \implies h = \sqrt{pq}$
(A)	$A_{\text{left}} = \frac{1}{2}ph \wedge A_{\text{right}} = \frac{1}{2}qh \implies A = A_{\text{left}} + A_{\text{right}} = \frac{1}{2}(p+q)\sqrt{pq}$
(PK)	$a^2 = h^2 + q^2 \wedge a^2 = (p+q)q \implies h^2 = (p+q)q - q^2 = pq \implies h = \sqrt{pq}$
(A)	$A = \frac{1}{2}ch = \frac{1}{2}(p+q)\sqrt{pq}$
	Result: $A \approx 29.10\,cm^2$

Problem c: $\{K, PH\}$

(K)	$a^2 = cq \implies c = \dfrac{a^2}{q} \implies p = \dfrac{a^2}{q} - q$
(PH)	$a^2 = h^2 + q^2 \wedge h^2 = pq \implies p = \dfrac{a^2 - q^2}{q}$
	<u>*Result:*</u> $p \approx 4.67\,cm$

Problem d: $\{KZ, HZ\}$

(KZ)



(HZ)

The length of the side of the square is about 4.27 cm.

Problem e: $\{PKTA, KHTA\}$

Let B_1 and B_2 be the tangential points of the tangents from P to the circle;

$\overline{PM} =: c$; $PM \cap B_1B_2 =: \{Q\}$; $\overline{B_1Q} = \overline{QB_2} =: h$; $\overline{PQ} =: p$; $\overline{QM} =: q$

It apparently holds: $\overline{MB_1} = \overline{MB_2} = r$

Further the property of tangents holds:

(T) $PB_1 \perp MB_1 \wedge PB_2 \perp MB_2$, i.e.:

$\triangle PB_1M$ and $\triangle PMB_2$ are rightangled.

(This property is presupposed for every solution way.)

(PK) $r^2 = q^2 + h^2 \wedge r^2 = cq \implies h^2 = r^2 - \frac{r^4}{c^2} \implies h = \frac{r}{c}\sqrt{c^2 - r^2}$

(A) $A = 2 \cdot \frac{1}{2}(c - q)h = (c - \frac{r^2}{c})\frac{r}{c}\sqrt{c^2 - r^2} = \frac{r}{c^2}(c^2 - r^2)\sqrt{c^2 - r^2}$

(KH) $r^2 = cq \wedge h^2 = pq \implies h^2 = (c - \frac{r^2}{c}) \cdot \frac{r^2}{c} \implies h = \frac{r}{c}\sqrt{c^2 - r^2}$

(A) as above

Set: $\overline{PB_1} = \overline{PB_2} =: b$

(PA) $c^2 = b^2 + r^2 \wedge A_{\triangle PMB_1} = \frac{1}{2}br \implies A_{\triangle PMB_1} = \frac{r}{2}\sqrt{c^2 - r^2}$

(PK) $r^2 = q^2 + h^2 \wedge r^2 = cq \implies h = \frac{r}{c}\sqrt{c^2 - r^2}$

(A) $A_{\triangle QMB_1} = \frac{1}{2}qh = \frac{1}{2}\frac{r^2}{c}\frac{r}{c}\sqrt{c^2 - r^2}$

$A = 2 \cdot (A_{\triangle PMB_1} - A_{\triangle QMB_1}) = \frac{r}{c^2}(c^2 - r^2)\sqrt{c^2 - r^2}$

Other solution ways are possible if the applied theorems are used repeatedly.

Result: $A \approx 9.94\, cm^2$

A.2 Solution patterns

The following Table 6 provides the protocols of the utilized solution ways in the geometry test. For each subject and each of the five evaluated problems the applied solution way is presented (as far as identifiable).

The following denotations are used in Table 6:

( ) The solution way is correct, but the numerical result is incorrect;

– The problem is not solved or not correctly solved;

* A non-modeled solution way or an additional solution way is applied.

Table 6: Solution patterns of the test

Subjects	Problem (current number in the test) a (4)	b (9)	c (5)	d (7)	e (13)	Solution pattern	Predicted yes(+)/no(−)
01	−	−	−	−	−	∅	+
02	H	−	−	−	−	a	+
03	−	−	−	−	−	∅	+
04	H	HA	PH	−	P∗TA	abce	+
05	−	−	−	−	−	∅	+
06	H	HA	K	KZ	P∗TA	abcde	+
07	H	−	−	−	−	a	+
08	H	HA	(PH)	−	(P∗TA)	ab(ce)	+ (+)
09	H	−	PH	−	PH∗TA	ace	−
10	−	−	−	−	−	∅	+
11	H	HA	PH	HZ	−	abcd	+
12	−	−	−	−	−	∅	+
13	H	HA	K	−	−	abc	+
14	−	−	−	−	−	∅	+
15	PK	−	−	−	−	a	+
16	H	HA	K	KZ	(P∗TA)	abcd(e)	+ (+)
17	−	−	−	−	−	∅	+
18	H	HA	PH	−	−	abc	+
20	−	(PKA)	∗	−	−	(b)c	+ (+)
22	H	HA	PH	HZ	PH∗TA	abcde	+
23	H	(HA)	−	−	−	a(b)	+ (+)
24	H	HA	K	−	KHTA	abce	+
26	H	HA	(PH)	HZ	P∗TA	ab(c)de	− (+)
30	H	HA, PKA	PH	HZ	P∗TA	abcde	+
31	H	(HA)	PH	−	P∗TA	a(b)ce	− (+)
32	−	−	−	−	−	∅	+
33	−	−	−	−	−	∅	+
34	−	∗A	P∗	−	P∗TA	bce	−
38	H	HA	K	−	−	abc	+
39	H	HA	−	−	−	ab	+
41	H	HA	PH	HZ	∗PTA	abcde	+
42	H	−	−	−	−	a	+
43	H	HA	−	−	−	ab	+
44	H	HA	P∗	−	−	abc	+
45	H	HA	PH	−	−	abc	+
46	∗	−	∗	−	−	ac	+
47	P∗	HA	K	HZ	PKTA	abcde	+
48	H	HA	PH	HZ	−	abcd	+
49	H	HA	PH	HZ	PKHTA	abcde	+
50	H	HA	PH	−	−	abc	+
51	H	PKA	K	−	PKHTA	abce	+
52	H	HA	PH	−	−	abc	+
53	H	HA	(PH)	−	−	ab(c)	+ (+)
54	H	HA	PH	−	P∗TA	abce	+
55	∗	P∗A	∗	−	P∗TA	abce	+
56	H	PHA	PH	−	PH∗TA	abce	+
58	H	HA	PH	−	(PKTA)	abc(e)	+ (+)
59	H	HA	K	−	PKHTA	abce	+
60	H	HA	PH	−	−	abc	+
61	H	HA	PH	−	−	abc	+
62	H	HA	PH	−	−	abc	+
63	H	HA	PH	−	−	abc	+

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